Mean Ruin Time for the Risk Reserve Process

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1. Introduction

In many companies they reserve the fund, which is said to be the risk reserve, to avoid the ruin incurred by the large-scale demand. This paper deals with such a risk reserve problem by using a mathematical method. First of all, we constitute the following mathematical model.

Let us denote by $X(t)$ the risk reserve level at time t, which has boundaries L and zero i.e. $0 \le X(t) \le L$. The reserve level is increasing with a rate $\alpha(x)$ decided by the present level $X(t) = x$ unless the process has the large-scale demand. We assume that the large-scale demand for the process occurs according to the Poisson process with parameter λ and the amount of each demand has a distribution $F(x)$ with the density $f(x)$ having the finite mean. Once the level reaches the upper boundary L, it remains at the level until the large scale demand occurs. We call $\{X(t)\}$ the risk reserve process. If the demand larger than the present level occurs, the reserve becomes empty, that is, the process is ruined. It is the most important problem to have the mean ruin time for the risk reserve process.

We can find the process in insurance and nursing insurance risk problems, mathematical finances, production-inventory problems, production systems with shocks, finite capacity queueing-inventory problem, M/G/1 queueing systems with removable server, and so on.

For the storage process with inflow and outflow, Doi and Ōsawa (1984) obtained the numerical solution of the probability concerning to the storage level and Doi (2000) obtained the mean ruin time. Doi, Nagai and Ōsawa (2002) obtained the ruin probability for the process $\{X(t)\}$.

The rest of the paper is organized as follows. In Section 2, we introduce the integro-difference equation to obtain the mean ruin time for the risk reserve process. In Section 3, we consider this equation and solve it. In Section 4, we deal with the case that the large-scale demand has an exponential distribution. In Section 5, we propose some numerical examples. Section 6 gives some remarks and further problems.

2. An Integro-Differential Equation

Let us define $t(x)$ as the mean ruin time incurred until the first epoch at which the reserve level drops down below the zero level given that the level is x at time 0.

We have the following equation with respect to $t(x)$.

$$t(x) = (1-\lambda\Delta t)t(x-\Delta x) + \Delta t + \lambda\Delta t \int_0^{L-x} t(x+y)dF(y) + o(\Delta t)\,. \tag{1}$$

From this equation the following difference equation is obtained.

$$t(x) - t(x-\Delta x) = -\lambda\Delta t \cdot t(x-\Delta x) + \Delta t + \lambda\Delta t \int_0^{L-x} t(x+y)dF(y) + o(\Delta t)\,. \tag{2}$$

Divided by Δx, the equation above is reduced to

$$\frac{t(x)-t(x-\Delta x)}{\Delta x} = -\lambda\frac{\Delta t}{\Delta x}\cdot t(x-\Delta x) + \frac{\Delta t}{\Delta x} + \lambda\frac{\Delta t}{\Delta x}\int_0^{L-x} t(x+y)dF(y) + \frac{o(\Delta t)}{\Delta x}\,. \tag{3}$$

Now note that

$$\alpha(x) = \frac{\Delta x}{\Delta t}$$

and let $\Delta x \to 0$.

Thus we constitute the following integro-differential equation.

$$\frac{dt(x)}{dx} = -\frac{\lambda}{\alpha(x)}t(x) + \frac{1}{\alpha(x)} + \frac{\lambda}{\alpha(x)}\int_0^{L-x} t(x+y)dF(y) \qquad (0 < x < L). \tag{4}$$

To evaluate the integral of the right hand side of (4) we use the following relation.

$$\begin{aligned}\frac{d}{dx}\int_0^{L-x} t(x+y)\{1-F(y)\}dy = {}& -t(L)\{1-F(L-x)\} \\ & + \int_0^{L-x} t'(x+y)\{1-F(y)\}dy\,.\end{aligned} \tag{5}$$

Transform $x+y$ into u then we have

$$\begin{aligned}&\frac{d}{dx}\int_0^{L-x} t(x+y)\{1-F(y)\}dy = -t(L-0)\{1-F(L-x)\} + \int_x^L t'(u)\{1-F(u-x)\}du \\ &= -t(L-0)\{1-F(L-x)\} + t(L-0)\{1-F(L-x)\} - t(x) + \int_x^L t(u)f(u-x)du \\ &= -t(x) + \int_x^L t(u)f(u-x)du\,.\end{aligned}$$

After transforming u into $x+y$, we have

$$\frac{d}{dx}\int_0^{L-x} t(x+y)\{1-F(y)\}dy = -t(x) + \int_0^{L-x} t(x+y)f(y)dy$$

$$= -t(x) + \int_0^{L-x} t(x+y)dF(y)\,.$$

From (4) and (5) we have

$$\frac{dt(x)}{dx} = \frac{\lambda}{\alpha(x)}\left\{\frac{d}{dx}\int_0^{L-x} t(x+y)\{1-F(y)\}dy\right\} + \frac{1}{\alpha(x)}\,, (0 < x < L). \tag{6}$$

For convenience, we take the increasing rate as follows :

$$\alpha(x) = \alpha\ \ (0 < x \leq L)\,.$$

Especially we denote by t_L the mean ruin time if $x = L$. Then the boundary condition is the following.

$$\lambda t_L \quad = \quad \alpha t(L-)\,. \tag{7}$$

By integration of (6), in the case of $\alpha(x) = \alpha$, from x to L, we have the following integral equation.

$$t(x) = t(L-) - \frac{1}{\alpha}(L-x) + \frac{\lambda}{\alpha}\int_0^{L-x} t(x+y)\{1-F(y)\}dy\,. \tag{8}$$

In the next section we constitute the renewal equation from (8).

3. A Renewal Equation and its Solution

In this section we consider the renewal equation to solve (8) with boundary condition (7).

Let us define the functions :

$$a(x) = t(L-) - \frac{1}{\alpha}(L-x), \tag{9}$$

$$h(x) = \frac{\lambda}{\alpha}\{1-F(x)\}\,. \tag{10}$$

By use of these functions we have a simple form of (8) as follows :

$$t(x) = a(x) + \int_0^{L-x} t(x+y)h(y)dy,\ \ (0 < x < L)\,. \tag{11}$$

Since this equation is a defective renewal equation, we define a number δ as the unique solution to the equation

$$\int_0^{\infty} e^{-\delta y}h(y)dy = 1\,. \tag{12}$$

Now define a probability density function using δ as follows :

$$h_0(x) = \frac{\lambda}{\alpha} e^{-\delta x}\{1 - F(x)\}, \quad (0 < x < L)\,.$$

We confirm the following relation from (10) and (12).

$$\int_0^\infty h_0(x)dx = 1\,.$$

Furthermore we define

$$t_0(x) = e^{\delta x} t(x),\ a_0(x) = e^{\delta x} a(x),\ H_0(y) = \int_0^y h_0(t)dt \quad (0 < x < L)\,,$$

where $H_0(y)$ is the distribution function. Thus we have a standard renewal equation

$$t_0(x) = a_0(x) + \int_0^{L-x} t_0(x+y)dH_0(y) \quad (0 < x < L)\,. \tag{13}$$

We proceed to get the solution of (13). Substituting $t_0(x)$ to the right hand side of (13) recursively, we get

$$t_0(x) = a_0(x) + \int_0^{L-x} a_0(x+y)dH_0(y) + \int_0^{L-x}\int_0^{L-(x+y)} t_0(x+y+z)dH_0(z)dH_0(y)\,.$$

Transform $y + z$ into u in the right hand side above. Then we have

$$\begin{aligned}\int_0^{L-x}\int_0^{L-(x+y)} t_0(x+y+z)dH_0(z)dH_0(y) &= \int_0^{L-x}\int_y^{L-x} t_0(x+u)dH_0(u-y)dH_0(y)\\ &= \int_0^{L-x} t_0(x+u)\int_0^u dH_0(u-y)dH_0(u)\\ &= \int_0^{L-x} t_0(x+u)dH_0^{2*}(u)\end{aligned}$$

where H_0^{2*}(u) is the convolution of $H_0(u)$ with itself. After n-th recursive substitution we have

$$t_0(x) = a_0(x) + \sum_{k=1}^{n-1}\int_0^{L-x} a_0(x+y)dH_0^{k*}(y) + \int_0^{L-x} t_0(x+y)dH_0^{n*}(y)\,.$$

where $H_0^{n*}(u)$ is the n-th fold convolution of $H_0(u)$ with itself. Furthermore we have

$$\lim_{n\to\infty}\int_0^{L-x} t_0(x+y)dH_0^{n*}(y) = 0\,,$$

because

$$\left|\int_0^{L-x} t_0(x+y)dH_0^{k*}(y)\right| \le K\int_0^{L-x} dH_0^{k*}(k) = KH_0^{k*}(L-x) < K\beta^k \ (K > 0, 0 < \beta < 1)$$

and

$$\sum_{k=1}^{\infty}\left|\int_0^{L-x} t_0(x+y)dH_0^{k*}(y)\right| < K\sum_{k=1}^{\infty}\beta^k = \frac{K}{1-\beta}$$

implies that

$$\sum_{k=1}^{\infty}\int_0^{L-x} t_0(x+y)dH_0^{k*}(y)$$

converges. It is well known that the general term converges to 0 if the infinite series converges absolutely. Let us denote the renewal function corresponding to the $H_0(y)$ by $M(y)$:

$$M(y) = \sum_{n=1}^{\infty} H_0^{n*}(y)\,. \tag{14}$$

The solution of the standard renewal equation (13) is

$$t_0(x) = a_0(x) + \int_0^{L-x} a_0(x+y)dM(y)\,.$$

Since

$$t(x) = e^{-\delta x}t_0(x),\; a(x+y) = e^{-\delta(x+y)}t_0(x+y),$$

we obtain the solution of (11) as follows :

$$t(x) = a(x) + \int_0^{L-x} e^{\delta y}a(x+y)dM(y)\,. \tag{15}$$

In the next section we deal with the case that the large-scale demand has an exponential distribution.

4. A Case of Exponential Large Scale Demand

Let the amount of large-scale demand have an exponential distribution with parameter μ. From (14), we have

$$M(y) = \int_0^y \sum_{k=1}^{\infty} H_0^{(k-1)*}(y-u)\frac{\lambda}{\alpha}e^{-\frac{\lambda}{\alpha}u}du = \frac{\lambda}{\alpha}\int_0^y M(y-u)e^{-\frac{\lambda}{\alpha}u}du + 1 - e^{-\frac{\lambda}{\alpha}x}. \tag{16}$$

Let us denote by $m(s)$ Laplace Transform of $M(y)$. Then we make the following evaluation. For Laplace Transform of the first term of the right hand side of (16) we evaluate :

$$\begin{aligned}\mathfrak{L}\left[\int_0^y M(y-u)e^{-\frac{\lambda}{\alpha}u}du\right] &= \int_0^{\infty} M(u)e^{-\frac{\lambda}{\alpha}u}\int_u^{\infty} e^{-(s+\frac{\lambda}{\alpha})y}dydu \\ &= \frac{\alpha}{\alpha s+\lambda}m(s)\,.\end{aligned}$$

Then we have

$$m(s) = \frac{\lambda}{\alpha s^2}\,.$$

Finally, by use of Inverse Transform, we have

$$M(y) = \frac{\lambda}{\alpha} y\,. \tag{17}$$

By use of (9) and (17), (15) is reduced to

$$t(x) = \frac{\lambda}{\alpha} t_L - \frac{1}{\alpha}\Big(1 - \frac{\lambda}{\alpha\delta}\Big)(L-x) + \frac{\lambda}{\alpha\delta}\Big(\frac{\lambda}{\alpha} t_L - \frac{1}{\alpha\delta}\Big)(e^{\delta(L-x)} - 1)\,.$$

Furthermore δ is given by (12) as

$$\delta = \frac{\lambda}{\alpha} - \mu\,.$$

Finally, we obtain the mean ruin time as follows :

$$t(x) = \frac{\lambda}{\alpha} t_L + \frac{1}{\alpha}\Big(\frac{\lambda}{\lambda - \alpha\mu} - 1\Big)(L-x) + \frac{\lambda}{\lambda - \alpha\mu}\Big(\frac{\lambda}{\alpha} t_L - \frac{1}{\lambda - \alpha\mu}\Big)(e^{(\frac{\lambda}{\alpha}-\mu)(L-x)} - 1)\,. \tag{18}$$

5. Numerical Examples

We consider the case that the upper boundary $L = 10$, and the mean amount of large scale demand $1/\mu = 10$. Usually we take $t_L = 1.0$.

In Japan, we could find the mean time interval of occurrence of large-scale demands such as earth quakes : $1/\lambda = 1.429$ (year). If we have the increasing rate $\alpha = 3.0, 4.0$, we obtain the following table.

x	7.5	8.0	8.5	9.0	9.5	10.0
$t(x)\,(\alpha = 3.0)$	0.153	0.324	0.495	0.665	0.834	1.0
$t(x)\,(\alpha = 4.0)$	0.367	0.495	0.622	0.749	0.876	1.0

We can improve the mean ruin time, given that the initial reserve level $x = 8.0$, from 0.324 to 0.495 as α increases from 3.0 to 4.0. We can see the effect of increasing α (33% up) makes the improving of mean ruin time (53% up).

6. Some Comments and Further Result

For the Gamma distribution with parameters $(2, \mu)$ we have $M(y)$ as follows :

$$M(y)=\int_0^y\sum_{k=1}^{\infty}H^{(k-1)*}(y-u)\mu^2ue^{-\mu u}du=\mu^2\int_0^y\{M(y-u)+1\}ue^{-\mu u}du\,. \tag{19}$$

Taking Laplace Transform for (19), we have

$$m(s)=\mu^2\{\frac{1}{s}\cdot\frac{1}{(s+\mu)^2-\mu^2}\}\,.$$

By use of Inverse Transform we have

$$M(y)=\frac{1}{4}(2\mu y+e^{-2\mu y}-2\mu)\,.$$

In the same way of exponential case we can get the mean ruin time in the case of Gamma distribution.

In this paper we consider the risk reserve process with increasing phase. For the case with two phases which have increasing and decreasing rates, we have just had the ruin probability analytically, Doi (2006), as follows :

$$P_0=\frac{\lambda P_L}{\nu_0}\left[\int_0^L\{I_{(l,L)}(y)p^*(0,y)+p^*(1,y)\}\{1-F(y)\}dy+1-F(L)\right]\,,$$

where

$$p^*(0,y)=\frac{p(0,y)}{P_L}\,,p^*(1,y)=\frac{p(1,y)}{P_L}\text{ and }P_0^*=\frac{P_0}{P_L}\,.$$

($p(0,y)$: the probability of the reserve level y given that the phase is outflow; $p(1,y)$: that of the case of inflow; P_L : that of the case of full reserve)
We have $p(0,y),p(1,y)$ as follows :

$$p(0,y)=B(y)+\int_0^{L-y}e^{\delta_0 u}B(y+u)dM_0(u)$$

where

$$B(y)=A_0(y)-\frac{\lambda}{\alpha_0}\int_0^{L-y}A_0(y+u)\{1-F(u)\}du\,,$$

$$A_0(y)=\frac{1}{\alpha_0}\{\nu_L+\lambda F(L-y)\}P_L\,,$$

$$M_0(u)=\sum_{n=1}^{\infty}H_0^{2n*}(u),H_0(u)=\frac{\lambda}{\alpha_0}\int_0^u e^{-\delta_0 v}\{1-F(v)\}dv$$

and δ_0 is the unique solution of the equation

$$\int_0^\infty \frac{\lambda}{\alpha_0} e^{-\delta_0 x}\{1-F(x)\}dx = 1.$$

($H_0^{2n*}(y)$ is the $2n$-th fold convolution of $H_0(y)$.)
For $l < x < L$ then $p(1,x)$ is obtained as follows :

$$p(1,y) = \frac{\nu_L + \lambda}{\alpha_1}\{1 + \int_0^{L-y} e^{\delta_1 u} dM_1(u)\}P_L\,,$$

where

$$M_1(u) = \sum_{n=1}^{\infty} H_1^{n*}(u), H_1(u) = \frac{\lambda}{\alpha_1}\int_0^u e^{-\delta_1 v}\{1-F(v)\}dv$$

and δ_1 is the unique solution of the equation

$$\int_0^\infty \frac{\lambda}{\alpha_1} e^{-\delta_1 x}\{1-F(x)\}dx = 1.$$

For $0 < x < l$ then $p(1,y)$ is obtained as follows:

$$p(1,y) = A_1(y) + \int_0^{l-y} e^{\delta_2 u} A_1(y+u) dM_2(u)\,,$$

where

$$\begin{aligned} A_1(y) = p(1,l-) &- \frac{\lambda}{\alpha_2}P_L\{F(L-l) - F(L-y)\} \\ &- \frac{\lambda}{\alpha_2}\int_y^l \int_l^L \{p(0,u) + p(1,u)\}dF(u-v)dv\,, \end{aligned}$$

$$M_2(u) = \sum_{n=1}^{\infty} H_2^{n*}(u), H_2(u) = \frac{\lambda}{\alpha_2}\int_0^u e^{-\delta_2 v}\{1-F(v)\}dv$$

and δ_2 is the unique solution of the equation

$$\int_0^\infty \frac{\lambda}{\alpha_2} e^{-\delta_2 x}\{1-F(x)\}dx = 1.$$

In this paper, we considered the mean ruin time in steady state, however we need to have one in finite time. Furthermore, we need to have the ruin probability in finite time.

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