# Queues with Random Arrival Acceptance Windows

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## 1. Introduction

We consider a queueing system with a random arrival acceptance window (*RAAW*). An ordinary simple queueing system consists of a server that processes requests of customers, and a waiting line (a queue) where customers have to wait before receiving services. Moreover, there is an arrival stream of customers to the system. In this queue, the terrible congestion may occur depending on the stream of customers. Preparing in such cases, in the queue with RAAW, the system controls the flow of arriving customers and congestion in the queue, by opening and closing the window alternately. First of all, we start with the mathematical model of this queue.

### 1. 1 Mathematical Model

In an ordinary queue, the stream of customers is described as an arrival process and the aspect of services that the server processes according to requests of customers is referred to a service process. The simple single server queue is termed the A/S/1 queue, where A refers to the form of the arrival time distribution, S to the form of the service time distribution, and the "1" indicates the single server. This is well known as the Kendall's notation.

On the other hand, in a single server queue with RAAW, the opening and closing of a window is alternately repeated. While the window is open, the system can accept at most $m$ customers, where $m$ is a positive integer. Let $t_n$ be the instance when the $n$ th window is open, then the duration $T_n = t_{n+1} - t_n$ is called the $n$ th cycle time, where $0 \le t_1 < t_2 < \cdots$.

Let $\tau_{nk}$ be the actual arrival instance of the $k$ th customer, for $k = 1, 2, \cdots, m$, during the time interval $[t_n, t_{n+1})$, where $t_n \le \tau_{n1} < \tau_{n2} < \cdots < \tau_{nm}$. If $\tau_{nm} < t_{n+1}$, then the $m$ th customer during $[t_n, t_{n+1})$ is accepted to the system and the window is closed at $\tau_{nm}$. If $\tau_{nm} \ge t_{n+1}$, then the window is closed just before $t_{n+1}$ and the next window immediately opens at $t_{n+1}$. While the window is closed, the system can accept no customers. That is, if $\tau_{nk} < t_{n+1} \le \tau_{nk+1}$, only $k$ customers are accepted to the system. Through these two periods, customers accepted to the system are served continuously.

This queue is denoted by a string of the type A|B$^m$/S/1, where A refers to the form of the cycle time distribution, B to the form of the arrival time distribution, S to the form of the service time distribution. In particular, if $m = 1$, it is merely written as A|B/S/1.

A typical example of the queue with RAAW is a scheduled arrival system, like a dental appointment system in Japan. In this system, a customer (patient) who should receive treatment will be scheduled his or her arrival time. If he or she would not enter the system by the appointment time, then he or she could not receive the service. Moreover, we can observe queues with RAAW in practical situation, for example, production systems, business processes and so on.

### 1. 2 Outline of Paper

This paper is essentially comprised of two problems on the queue with RAAW.

We first investigate the system state probabilities and derive the stationary distributions of the queue-length and the waiting-time processes in the next two sections.

Let $Q(t)$ be the number of customers in the system at time $t$. We now choose $t_n$ as the imbedded points for the queue-length process $\{Q(t); t \geq 0\}$ and define the imbedded process $\{Q_n; n \geq 0\}$ where $Q_n = Q(t_n - 0)$. Let $W(t)$ be the total service time of customers in the system at time $t$, then the $\{W(t); t \geq 0\}$ is called the workload process. Let $W_{nk}$ be the waiting time of the $k$ th customer during $[t_n, t_{n+1})$, then it is described by

$$W_{n_k} = \begin{cases} [W(t_n) - (\tau_{n_k} - t_n)]^+, & t_n \leq \tau_{n_k} < t_{n+1}, \\ 0, & \text{otherwise}, \end{cases}$$

where $[x]^+ = \max(x, 0)$.

Section 2 deals with M|M/M/1 queue, where M denotes an exponential distribution. That is, the durations $T_n$ are independent and common exponentially distributed random variables, at most one customer can be accepted during $[t_n, t_{n+1})$, the delayed-arrival time of the customer, $\tau_{n1} - t_n$, has an exponential distribution, and service times are also exponential. Section 3 analyzes GI|GI/M/1 queue, where GI means the process is generally distributed. That is, the $\{T_n\}$ is a renewal process, at most one customer can be accepted during $[t_n, t_{n+1})$, the delayed-arrival time of the customer is generally distributed, and service times are exponential. In these two sections, we derive the stationary distributions of the queue-length and the waiting-time processes for each system.

We next consider the problem to determine the optimal window size in some sense. The system makes profit on services for accepted customers, but incurs the waiting cost for each accepted customer. Moreover, if a customer is not accepted, the system suffers an opportunity loss. Section 4 studies the optimal window size to maximize reward of the system for the GI|GI/M/1 queue. And it is determined by numerical treatment for some queues with RAAW.

Finally, in Section 5 we express further problems for queues with RAAW.

The queue with RAAW was first introduced by Doi, Chen and Ōsawa (1997). They considered GI|M/M/1 queue. Moreover, Ōsawa, Doi, Chen and Shima (2000) dealt with GI|GI/M/1 queue. We shall refer to their results in section 3. In these papers, stationary distributions of queue-length and waiting processes have been obtained. The problem to determine the optimal window size has been seen in Doi, Ōsawa and Chen (2002). In section 4, we define a slight different reward from one considered in their paper.

## 2. M|M/M/1 Queue

In this section, we study the M|M/M/1 queue. We assume that the cycle times $\{T_n\}$ are independent and common exponentially distributed random variables with finite mean $1/\lambda$. We also assume that delayed arrival times $\{\tau_{n1} - t_n\}$ are independent and common exponentially distributed random variables with finite mean $1/\nu$. Service times of accepted customers have an exponential distribution with parameter $\mu$. For the queue-length and the waiting-time process, we consider the necessary and sufficient condition that the limiting distribution of the processes exists and derive their stationary distributions.

### 2.1 Queue-Length Process

Consider the queue-length process $\{Q(t)\}$ of the M|M/M/1 queue. For the purpose, let $J(t)$ be the state of the window at time $t$, that is,

$$J(t) = \begin{cases} 1, & \text{if the window is open at time } t, \\ 0, & \text{if the window is closed at time } t. \end{cases}$$

The state of the process $\{(J(t), Q(t))\}$ is given by $(i, j)$, $i = 0, 1$, $j \geq 0$. Define the following matrices

$$\boldsymbol{A}_0 = \begin{pmatrix} 0 & 0 \\ \nu & 0 \end{pmatrix}, \qquad \boldsymbol{A}_1 = \begin{pmatrix} -(\lambda+\mu) & \lambda \\ 0 & -(\nu+\mu) \end{pmatrix},$$

$$\boldsymbol{A}_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}, \quad \text{and} \quad \boldsymbol{B}_{00} = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\nu \end{pmatrix}.$$

Then $\{(J(t), Q(t))\}$ is a quasi birth-death process with the transition rate matrix $\boldsymbol{Q}$

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{B}_{00} & \boldsymbol{A}_0 & \boldsymbol{0} & \cdot & \cdot & & \\ \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \boldsymbol{0} & \cdot & \cdot & \\ \boldsymbol{0} & \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \boldsymbol{0} & \cdot & \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_2 & \boldsymbol{A}_1 & \boldsymbol{A}_0 & \boldsymbol{0} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where $\boldsymbol{0}$ is a matrix of all zeros of the dimension 2.

Let $\pi_i(j)$ be the stationary probability of state $(i, j)$, then the probability vectors $\boldsymbol{\pi}(j) = (\pi_0(j), \pi_1(j))$ have a matrix geometric form, that is,

$$\boldsymbol{\pi}(j) = \boldsymbol{\pi}(1)\boldsymbol{R}^{j-1}, \quad j = 1, 2, \cdots,$$

where $\boldsymbol{R}$ is the minimal matrix satisfying that

$$\boldsymbol{A}_0 + \boldsymbol{R}\boldsymbol{A}_1 + \boldsymbol{R}^2\boldsymbol{A}_2 = \mathbf{0}.$$

Note that $\boldsymbol{R}$ is termed the rate matrix. Moreover, $\boldsymbol{\pi}(0)$ and $\boldsymbol{\pi}(1)$ satisfy that

$$(\boldsymbol{\pi}(0), \boldsymbol{\pi}(1)) \begin{pmatrix} \boldsymbol{B}_{00} & \boldsymbol{A}_0 \\ \boldsymbol{A}_2 & \boldsymbol{A}_1 + \boldsymbol{R}\boldsymbol{A}_2 \end{pmatrix} = \mathbf{0}.$$

To determine these probability vectors, we need the normalizing condition

$$\boldsymbol{\pi}(0)\boldsymbol{e} + \boldsymbol{\pi}(1)(\boldsymbol{I} - \boldsymbol{R})^{-1}\boldsymbol{e} = 1,$$

where $\boldsymbol{e}$ is a column vector of all 1s.

### 2. 2 Workload Process

We easily derive the stationary distribution of the workload process by using results in the previous subsection. Define $F(x) = \mathbf{P}[W(t) \leq x]$, then we have

$$F(x) = \sum_{j=0}^{\infty} \boldsymbol{\pi}(j)\boldsymbol{e}\mathbf{P}[\,W(t) \leq x \,|\, Q(t) = j\,].$$

For $x > 0$, $F(x)$ has the density $f(x)$ given by

$$f(x) = \sum_{j=1}^{\infty} \mu\,\boldsymbol{\pi}(1)\boldsymbol{R}^{j-1}\boldsymbol{e}\mu_{j-1}(x) = \mu\,\boldsymbol{\pi}(1)e^{-\mu(\boldsymbol{I}-\boldsymbol{R})x}\boldsymbol{e},$$

where $\mu_j(x) = \dfrac{e^{-\mu x}(\mu x)^j}{j!}$, $x \geq 0$, $j = 0, 1, 2, \cdots$. Since $F(0) = \boldsymbol{\pi}(0)\boldsymbol{e}$, we get

$$F(x) = 1 - \boldsymbol{\pi}(1)(\boldsymbol{I} - \boldsymbol{R})^{-1}e^{-\mu(\boldsymbol{I}-\boldsymbol{R})x}\boldsymbol{e}.$$

## 3. GI|GI/M/1 Queue

In this section, we deal with the GI|GI/M/1 queue. We assume that the cycle times $\{T_n\}$ are independent and identical distributed (*i.i.d.*) random variables with a distribution function $A(x)$ which has the finite mean $1/\lambda$. We also assume that delayed arrival times $\{\tau_{n1} - t_n\}$ are independent and identical distributed random variables with a distribution function $B(x)$. Then we should note the probability $p_A$ that a customer is accepted to the system during a cycle time is given by

$$p_A = \int_0^{\infty} B(y)\,dA(y). \tag{1}$$

$p_A$ is called as the acceptance probability. Service times of accepted customers assumed to be exponentially distributed with parameter $\mu$. For the queue-length and the waiting-time process, we consider the necessary and sufficient condition that the limiting distribution of the processes exists and derive their stationary distributions.

### 3. 1 Queue-Length Process

It is clear that the imbedded queue-length process $\{Q_n; n = 1, 2, \ldots\}$ constitutes a Markov chain. Define transition probabilities for this process

$$P_{ij} = \mathbf{P}[\, Q_{n+1} = j \,|\, Q_n = i\,], \quad i, j \geq 0,$$

we then have, for $j \geq 2$,

$$\begin{aligned} P_{ij} &= \int_0^\infty \mu_{i+1-j}(y) B(y)\, dA(y) + \int_0^\infty \mu_{i-j}(y) \overline{B}(y)\, dA(y), \quad i \geq j, \\ P_{j-1,j} &= \int_0^\infty e^{-\mu y} B(y)\, dA(y), \quad i = j-1. \end{aligned}$$

Moreover, for $j = 1$ and 0,

$$\begin{aligned} P_{i1} &= \int_0^\infty \overline{B}(y) \mu_{i-1}(y) dA(y) + \int_0^\infty \int_0^y B(t) \mu_{i-1}(t) \mu\, dt e^{-\mu(y-t)} dA(y) \\ &\quad + \int_0^\infty \int_0^y \int_0^\tau \mu_{i-1}(t) \mu dt dB(\tau) e^{-\mu(y-\tau)} dA(y), \quad i \geq 1, \\ P_{01} &= \int_0^\infty \int_0^y dB(\tau)\, e^{-\mu(y-\tau)}\, dA(y), \\ P_{i0} &= \int_0^\infty \int_0^y \mu \mu_{i-1}(t) dt \overline{B}(y) dA(y) + \int_0^\infty \int_0^y B(t) \mu_{i-1}(t) \mu dt (1 - e^{-\mu(y-t)}) dA(y) \\ &\quad + \int_0^\infty \int_0^y \int_0^\tau \mu_{i-1}(t) \mu dt dB(\tau)(1 - e^{-\mu(y-\tau)}) dA(y), \quad i \geq 1, \\ P_{00} &= \int_0^\infty \overline{B}(y) dA(y) + \int_0^\infty \int_0^y dB(\tau)(1 - e^{-\mu(y-\tau)})\, dA(y). \end{aligned}$$

For any other cases, i.e., $j \geq i + 2$, we have $P_{ij} = 0$.

Thus we obtain the transition probability matrix of the GI/M/1 type;

$$\boldsymbol{P} = \begin{pmatrix} P_{00} & P_{01} & 0 & \cdot & \cdot & & & \\ P_{10} & P_{11} & a_0 & 0 & \cdot & \cdot & & \\ P_{20} & P_{21} & a_1 & a_0 & 0 & \cdot & & \\ P_{30} & P_{31} & a_2 & a_1 & a_0 & 0 & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where $a_k$ is define by

$$a_k = \int_0^\infty \mu_k(y)B(y)\,dA(y) + \int_0^\infty \mu_{k-1}(y)\overline{B}(y)\,dA(y), \quad k \geq 1,$$

$$a_0 = \int_0^\infty e^{-\mu y} B(y)\,dA(y).$$

From the fundamental theory of the GI/M/1 type matrix, see Neuts (1981), we can see the behavior of the process $\{Q_n\}$. Denote the Laplace Stieltjes transform concerning the distribution $A(x)$ as $a[s]$;

$$a[s] = \int_0^\infty e^{-sx}\,dA(x), \quad s \geq 0.$$

Let $\Phi(z)$ be the generating function for a sequence $\{a_k, k \geq 0\}$, then we have

$$\Phi(z) = \sum_{k=0}^\infty a_k z^k = \int_0^\infty e^{-\mu(1-z)y}\{B(y) + z\overline{B}(y)\}\,dA(y), \qquad 0 \leq z \leq 1. \tag{2}$$

We then have the following. For the details, refer to Ōsawa, Doi, Chen and Shima (2000).

**Lemma 3. 1** *Let $\rho = \lambda/\mu$, then the equation $z = \Phi(z)$ has the unique solution $\zeta$ such that* $0 < \zeta < 1$ *if and only if* $\rho p_A < 1$.

**Theorem 3. 2** *The queue-length process $\{Q_n\}$ is positive recurrent if and only if $\rho p_A < 1$. Under this condition, the limiting distribution is given by*

$$\pi(k) = \lim_{n\to\infty} \mathbf{P}[Q_n = k] = \begin{cases} 1-\sigma, & k = 0, \\ \sigma(1-\zeta)\zeta^{k-1}, & k = 1, 2, \ldots, \end{cases}$$

*where*

$$\sigma = \frac{\beta[I]}{\displaystyle\int_0^\infty K(y)dA(y) + \beta[I] - \beta[K]},$$

*for distribution functions $K$ and $I$ defined by*

$$K(x) = 1 - e^{-\mu(1-\zeta)x}, \quad I(x) = 1, \quad x \geq 0.$$

*Moreover, for a distribution function $G$ on $[0, \infty)$, we here used a function*

$$\beta[G] = \int_0^\infty \int_0^y d(B \cdot G)(t)\ e^{-\mu(y-t)}\,dA(y).$$

From this theorem, we can obtain the mean queue-length $L$ and its variance $V_L$ as follows;

$$L = \sum_{k=0}^{\infty} k\pi(k) = \frac{\sigma}{1-\zeta}, \qquad V_L = \frac{\sigma(1+\zeta-\sigma)}{(1-\zeta)^2}.$$

### 3. 2 Queue-Length Process at Arrival Epochs

We study the queue-length distribution at actual arrival epochs in the steady state. Define the conditional probability $\pi(k\,|\,t)$ as follows;

$$\pi(k \mid t) = \mathbf{P}[\,Q(\tau_{n_1}-0) = k \,|\, t_n \le \tau_{n_1} < t_{n+1},\ \tau_{n_1} - t_n = t\,], \quad k = 0, 1, 2, \cdots.$$

Then $\{\pi(k\,|\,t),\ k \geq 0\}$ represents the queue-length distribution just before the actual arrival epoch of a customer whose delayed arrival time is $t$ during a cycle time interval. Using results in the previous subsection we can get

$$\begin{aligned}\pi(k \mid t) &= \sum_{i=k}^{\infty} \pi(i)\mu_{i-k}(t) = \sigma(1-\zeta)\zeta^{k-1}\, e^{-\mu(1-\zeta)t}, \qquad k = 1, 2, \cdots, \\ \pi(0 \mid t) &= \pi(0) + \sum_{i=1}^{\infty} \pi(i) \left(1 - \sum_{j=0}^{i-1} \mu_j(t)\right) = 1 - \sigma\, e^{-\mu(1-\zeta)t}.\end{aligned}$$

And its mean and variance are given by

$$L^*(t) = \frac{\sigma}{1-\zeta}\, e^{-\mu(1-\zeta)t}, \quad V_L^*(t) = \frac{\sigma(1+\zeta-\sigma)}{(1-\zeta)^2}\, e^{-\mu(1-\zeta)t}.$$

Moreover, we have the queue-length distribution $\{\pi^*(k),\ k \geq 0\}$ at actual arrival epochs as follows;

$$\begin{aligned}\pi^*(k) &= \frac{1}{p_A} \int_0^{\infty} \int_0^{y} \pi(k \mid t)\, dB(t)\, dA(y) = \eta(1-\zeta)\zeta^{k-1}, \quad k = 1, 2, \cdots, \\ \pi^*(0) &= \frac{1}{\displaystyle\int_0^{\infty} K(y)\, dA(y)} \int_0^{\infty} \int_0^{y} \pi(0 \mid t)\, dB(t)\, dA(y) = 1 - \eta,\end{aligned}$$

$$\text{where } \eta = \frac{\sigma}{\displaystyle\int_0^{\infty} K(y)\, dA(y)} \int_0^{\infty} \int_0^{y} \overline{K}(t)\, dB(t)\, dA(y). \tag{3}$$

Thus its mean and variance are given by

$$L^* = \frac{\eta}{1-\zeta}, \quad V_L^* = \frac{\eta(1+\zeta-\eta)}{(1-\zeta)^2}.$$

### 3. 3 Waiting Time

We define the conditional waiting time distribution of an accepted customer under that his or her delayed arrival time is $t$;

$$\begin{aligned} F^*(x|t) &= \mathbf{P}[\,W_{n_1} \le x \,|\, t_n \le \tau_{n_1} < t_{n+1},\, \tau_{n_1} - t_n = t\,], \\ F^*(x) &= \mathbf{P}[\,W_{n_1} \le x \,|\, t_n \le \tau_{n_1} < t_{n+1},\,]. \end{aligned}$$

Then we have

$$\begin{aligned} F^*(x|t) &= \sum_{k=0}^{\infty} \pi_k \mathbf{P}[\,W_{n_1} \le x \,|\, Q_n = k,\, t_n \le \tau_{n_1} < t_{n+1},\, \tau_{n_1} - t_n = t\,] \\ &= 1 - \sigma + \sum_{k=1}^{\infty} \sigma(1-\zeta)\zeta^{k-1} \Big\{ 1 - \sum_{i=0}^{k-1} \mu_i(t+x) \Big\} \\ &= 1 - \sigma e^{-\mu(1-\zeta)(t+x)}. \end{aligned}$$

Let $W^*(t)$ and $V_W^*(t)$ be the mean and variance of the conditional waiting time, respectively. Then we get

$$W^*(t) = \frac{\sigma}{\mu(1-\zeta)}\, e^{-\mu(1-\zeta)t}\,, \quad V_W^*(t) = \frac{\sigma(2-\sigma)}{\mu^2(1-\zeta)^2}\, e^{-\mu(1-\zeta)t}\,.$$

Moreover, we have the actual waiting time distribution as follows;

$$F^*(x) = \frac{1}{p_A} \int_0^{\infty} \int_0^{y} F^*(x|t)\, dB(t)\, dA(y) = 1 - \eta e^{-\mu(1-\zeta)x},$$

where $\eta$ is defined by (3). Hence the mean and the variance of the actual waiting time are given by

$$W^* = \frac{\eta}{\mu(1-\zeta)}\,, \quad V_W^* = \frac{\eta(2-\eta)}{\mu^2(1-\zeta)^2}\,.$$

## 4. Optimal Window Size

In this section, we deal with the optimal arrival acceptance window size for the GI|GI/M/1 queues with the deterministic or random arrival acceptance window. We introduce following costs to the system, that is, the system gets the profit by providing the customer service (the service profit), but incurs the waiting cost for each accepted customer. Moreover, if a customer is not accepted, the system suffers an opportunity loss. Let $\Lambda$ be the window size or the mean cycle time, i.e., $\Lambda = 1/\lambda$, then the reward that the system gains is dependent of $\Lambda$.

Denote the service profit and the waiting cost per unit time by $C_s$ and $C_w$, respectively. Let $C_\ell$ be the amount of damages by an opportunity loss of a customer. We then define an expected reward $r(\Lambda)$ per a customer or a cycle time;

$$r(\Lambda) = p_A \frac{1}{\mu} C_s - (1 - p_A) C_\ell - W^* C_w.$$

Let $T_c$ be the total hours of operation for the system, then the expected number opening RAAW is $\lambda T_c$. Thus the total expected reward $R(\Lambda)$, called as the reward function, is given by

$$R(\Lambda) = \lambda T_c r(\Lambda).$$

Using the results in the previous section, we consider the optimality of the RAAW size to maximize this reward function.

### 4. 1 Some Systems

Consider the queue of the D|GI or M|GI type, where D means a deterministic distribution. In this subsection, we deal with the following four cases; D|M/M/1 queue, D|D/M/1 queue, M|M/M/1 queue and M|D/M/1 queue.

(4.1) D|M/M/1 queue

In this queue, the delayed arrival time of a customer is exponentially distributed with mean $1/\nu$. From (1), the acceptance probability for a customer is obtained by

$$p_A = 1 - e^{-\nu/\lambda}.$$

$\Phi(z)$ defined by (2) is

$$\Phi(z) = e^{-\mu(1-z)/\lambda} \left\{1 - (1 - z)e^{-\nu/\lambda}\right\}.$$

Under $\rho p_A < 1$, we thus have the solution $\zeta$ of the equation $z = \Phi(z)$, $0 < z < 1$. And $\sigma$ in Theorem 3.2 is given by

$$\sigma = \frac{\nu(\nu - \mu\zeta)}{\kappa(\nu - \mu)} (e^{-\mu/\lambda} - e^{-\nu/\lambda})$$

where

$$\kappa = (\nu - \mu\zeta)\{1 - e^{-\mu(1-\zeta)/\lambda}\} + \nu\{1 - \mu(1 - \zeta)\}\{e^{-\mu/\lambda} - e^{-(\nu+\mu(1-\zeta))/\lambda}\}.$$

Then $\eta$ defined by (3) is reduced to

$$\eta = \frac{\sigma\nu\{1 - e^{-(\nu+\mu(1-\zeta))/\lambda)}\}}{\mu\{\nu + \mu(1 - \zeta)\}\{1 - e^{-\mu(1-\zeta)/\lambda}\}}.$$

From these values, we can get the stationary distributions given in section 3 and observe the optimal window size numerically.

(4.2) D|D/M/1 queue

Consider a queue with a deterministic delayed time with the mass at $1/\nu$, where $\lambda < \nu$. Then we have $p_A = 1$ and

$\Phi(z) = e^{-\mu(1-z)/\lambda}$.

(4.3) M|M/M/1 queue

Consider a queue with an exponential delayed time with mean $1/\nu$, then we have

$$p_A = \frac{\nu}{\lambda + \nu}.$$

Moreover, the equation $z = \Phi(z)$ is reduced to

$$\mu^2 z^2 - \mu(\lambda + \nu + \mu)z + \lambda\nu = 0.$$

Thus we get

$$\zeta = \frac{1}{2\mu}\left\{\lambda + \nu + \mu - \sqrt{(\lambda + \nu + \mu)^2 - 4\lambda\nu}\right\}.$$

(4.4) M|D/M/1 queue

Consider a queue with a deterministic delayed time with mass at $1/\nu$, then we have

$$p_A = e^{-\lambda/\nu}.$$

Moreover, the equation $z = \Phi(z)$ is reduced to

$$\mu z - \lambda e^{-\{\lambda + \mu(1-z)\}/\nu} = 0.$$

In these three cases, it is possible to consider the optimal window size in a similar way to case (4.1).

## 4. 2 Numerical Examples

We now consider numerical treatments for some systems in the previous subsection. Let $\mu$ = 1.0, that is, the mean service time assumed to be denoted by unit time, and let the total hours of operation be $T_c$ = 150. Moreover, assume that $C_s$ = 30, $C_w$ = 2 and $C_\ell$ = 10. In each case, we evaluate zeta, $\sigma$, eta and also $W^*$. Thus we can observe the movement of $R(\Lambda)$ and get the optimal window size.

For the D|M/M/1 queue with the mean delayed time $1/\nu$ = 0.25, the values of the acceptance probability $p_A$, the mean waiting times $W^*$ and the reward function $R(\Lambda)$ are presented in Table 1. The graph of $R(\Lambda)$ is given in Figure 1. We can observe here that $p_A$ is increasing and $W^*$ is decreasing to variations of the length $\Lambda = 1/\lambda$ of the window. On the other hand, $R(\Lambda)$ has only one maximal point as shown in Figure 1. Therefore, the optimal window size $\Lambda_{opt}$ is given by $\Lambda_{opt}$ = 1.3976.

Table 2 shows values of $p_A$, $W^*$ and $R(\Lambda)$ for the D|D/M/1 queue with the mean delayed time $1/\nu$ = 0.25. The graph of $R(\Lambda)$ is also given in Figure 2. It is observed that variations of these values have the same tendencies as the above case and the optimal window size is given

**Table 1. Values of $p_A$, $W^*$ and $R(\Lambda)$ for the D|M/M/1 with $1/\nu = 0.25$**

| $\Lambda$ | $p_A$ | $W^*$ | $R(\Lambda)$ |
|---|---|---|---|
| 1.300 | 0.9945 | 3.2527 | 2685.455 |
| 1.340 | 0.9953 | 2.5870 | 2757.986 |
| 1.380 | 0.9960 | 2.1090 | 2784.967 |
| 1.390 | 0.9962 | 2.0106 | 2786.858 |
| 1.395 | 0.9962 | 1.9640 | 2787.222 |
| 1.3976 | 0.9963 | 1.9404 | 2787.268 |
| 1.398 | 0.9963 | 1.9368 | 2787.267 |
| 1.400 | 0.9963 | 1.9190 | 2787.228 |
| 1.420 | 0.9966 | 1.7539 | 2784.056 |
| 1.460 | 0.9971 | 1.4824 | 2765.636 |
| 1.500 | 0.9975 | 1.2700 | 2736.088 |

**Table 2. Values of $p_A$, $W^*$ and $R(\Lambda)$ for the D|M/M/1 with $1/\nu = 0.25$**

| $\Lambda$ | $p_A$ | $W^*$ | $R(\Lambda)$ |
|---|---|---|---|
| 1.200 | 1.00 | 3.6336 | 2841.601 |
| 1.240 | 1.00 | 2.6150 | 2996.365 |
| 1.280 | 1.00 | 1.9875 | 3049.811 |
| 1.300 | 1.00 | 1.7597 | 3055.445 |
| 1.301 | 1.00 | 1.7495 | 3055.464 |
| 1.3013 | 1.00 | 1.7464 | 3055.466 |
| 1.302 | 1.00 | 1.7393 | 3055.461 |
| 1.310 | 1.00 | 1.6612 | 3054.676 |
| 1.320 | 1.00 | 1.5714 | 3051.961 |
| 1.360 | 1.00 | 1.2800 | 3026.473 |
| 1.400 | 1.00 | 1.0671 | 2985.615 |

**Figure 1. Graph of $R(\Lambda)$ for the D|M/M/1 with $1/\nu = 0.25$**

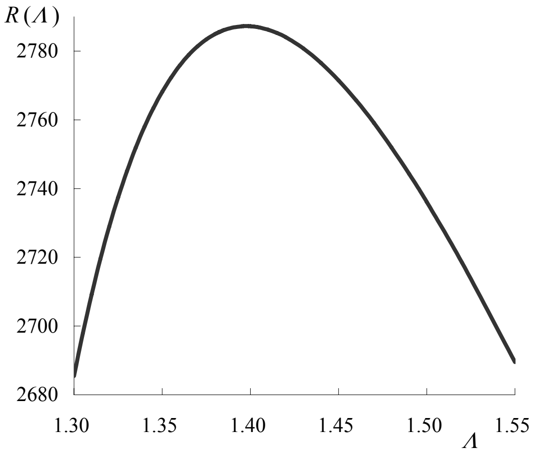


**Figure 2. Graph of $R(\Lambda)$ for the D|M/M/1 with $1/\nu = 0.25$**

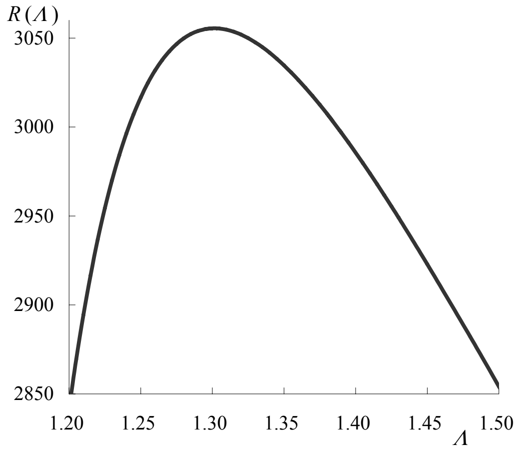


**Table 3. Optimal Window Size $\Lambda_{opt}$, for each queue with $1/\nu = 0.25$ or $1/\nu = 0.5$**

| Case | | $\Lambda_{opt}$ | $p_{Aopt}$ | $W^*_{opt}$ | max $R(\Lambda)$ |
|---|---|---|---|---|---|
| $\nu = 4$ | D\|M | 1.3976 | 0.996 | 1.9404 | 2787.268 |
| | D\|D | 1.3013 | 1.000 | 1.7464 | 3055.466 |
| | M\|M | 1.3122 | 0.840 | 1.6549 | 2319.264 |
| | M\|D | 1.1330 | 0.802 | 1.2123 | 2602.187 |
| $\nu = 2$ | D\|M | 1.4377 | 0.944 | 1.6300 | 2554.517 |
| | D\|D | 1.3233 | 1.000 | 1.7731 | 2998.623 |
| | M\|M | 1.2141 | 0.708 | 1.2886 | 1946.497 |
| | M\|D | 1.0471 | 0.620 | 0.5242 | 1971.824 |

by $\Lambda_{opt} = 1.3013$.

For the other queues considered in the previous section, the same properties are observed. For each queue with $1/\nu = 0.25$ and $1/\nu = 0.5$, Table 3 shows the optimal window size $\Lambda_{opt}$, and the acceptance probability $p_{Aopt}$, the mean waiting time $W^*_{opt}$ and the maximum of the reward $R(\Lambda)$. It is observed that the queue with the deterministic acceptance window may be more efficient than one with the random acceptance window.

## 5. Further Problems

The research of queues with RAAW still is a step in the early stages and further development will be needed. In particular, the following two problems seem to be important, that is, "development of models" and "comparison with the ordinary queue".

(I) Development of Models

Recently, Shin (2003) has analyzed the queue-length and waiting time processes in the GI|M/M$^m$/$c$ queue, using the matrix geometric method. It is desired that the analysis for the G|G/G type will be well done. Furthermore, it is important to obtain relations among the distributions at opening epochs of windows, arrival epochs of customers, closing epochs of windows and arbitrary time.

From the other point of view, there is a great variety of setting the cycle times. For example, following cases should be considered; both lengths of open and close periods of the window may be independently determined, or close period is deterministic, and so on. It is also important to determine the most efficient one of these systems.

(II) Comparisons with the Ordinary Queue

To make a comparison between the queue with RAAW and the ordinary queue is an important problem. For the purpose, we may need to study the divisibility of each process into two or more than two independent elements, the process in the ordinary queue and other processes. The latter processes given here can be taken with the one which shows the effect of the windows. Therefore, applying this divisibility, it is expected that the effect of the windows is quantitatively evaluated.

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