Bayesian Estimation of Stochastic Volatility Model Using Hamiltonian Monte Carlo Method and its Application to Nikkei 225 Data

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1. Introduction

Interest in time series data of returns on risky assets such as stocks, stock indices, and foreign exchange rates focuses on the variation of the second-order moment called volatility. In particular, stochastic volatility (SV) models are widely used in the analysis of the data of returns on risky assets[1)]. However, the inclusion of latent variables in an SV model makes it difficult to estimate the parameters of the model by maximum likelihood, so alternative estimation methods are needed. To meet this need, many previous studies have used Bayesian estimation based on Markov chain Monte Carlo (MCMC) methods[2)]. In an SV model, it is necessary to simultaneously sample the parameters describing the model and the latent variable, volatility, from the posterior distribution. Therefore, the number of volatility data becomes the same as the number of observations, so it is important to sample efficiently.

The Metropolis-Hastings (M-H) method and the Gibbs sampling method have been used as MCMC methods in previous studies using SV models. However, these MCMC methods have, among their problems, a small rejection rate when many random variables are estimated at once, so Takaishi (2008, 2009, 2013) proposed Bayesian estimation using the hybrid Monte Carlo method in the case of SV model estimation. Nugroho and Morimoto (2015) conducted an empirical analysis of the Tokyo Stock Price Index and Standard & Poor's 500 stock index using an SV model with the Riemann manifold Hamiltonian Monte Carlo method, which is a modification of the hybrid Monte Carlo method. In the present study, we propose estimating the parameters of an SV model by Bayesian estimation using the Hamiltonian Monte Carlo (HMC) method[3)].

The HMC method uses the molecular dynamics method developed in physics to update the parameters of the statistical model, and the M-H method is used to adopt or reject the updated parameters. The molecular dynamics method is used to numerically estimate the motion of interacting molecules and atoms based on classical mechanics such as Newtonian mechanics and analytical mechanics[4)].

In this paper, we explain the SV with leverage (SVL) model[5)], which is an extension of the basic SV model that uses not only the normal distribution but also the t-distribution for the distribution of the error term. In addition, we show an example of applying these models to the returns data of the Nikkei 225. As in previous studies, we found

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that the model provides a fit better with a t-distribution than with a normal distribution in terms of the existence of the leverage effect and the error distribution. Furthermore, we verified that Bayesian estimation by the HMC method is effective for the SV model.

The remainder of this paper is organized as follows: Section 2 describes the continuous-time SV models and discrete approximations used in this study. Section 3 explains Bayesian inference with the HMC method for SV models. Section 4 introduces the SV model with asymmetry and the SV model with a fat-tails distribution. Section 5 describes the data and the empirical results for the Nikkei 225 used in the analysis and discusses their implications. Section 6 presents conclusions.

2. Continuous-time SV models and discrete approximations

2.1 Continuous-time SV model

In the continuous-time SV model, the price of a risky asset is lognormally distributed and described by the following geometric Brownian motion model.

$$dS=\mu Sdt+\sigma Sdz_1. \tag{2.1}$$

Here, μ is a drift term, dt is a small change in time, σ is the standard deviation, and dz_1 is the Wiener process. In finance theory, σ is called volatility, whereas in financial econometrics, often σ^2 is called volatility. In the present study, we follow the latter and define σ^2 as the volatility. The variation of σ^2 is formulated as a continuous-time process as follows:

$$d\sigma^2=\psi\sigma^2dt+\delta\sigma^2dz_2. \tag{2.2}$$

Here, the Wiener process dz_2 may or may not be correlated with dz_1 in (2.1), corresponding to $\rho\neq 0$ and $\rho=0$, respectively. In addition, most studies of continuous-time SV models for risky asset prices have formulated the variation of σ^2 as the following continuous-time stochastic process with the mean-reverting property like the Ornstein-Uhlenbeck process:

$$d\sigma^2=\kappa[\theta-\sigma^2]dt+\delta\sigma dz_2. \tag{2.3}$$

Here, θ is the long-run mean and κ is the speed of long-run mean reversion. Thus, σ^2 reverts towards long-term mean θ at speed κ.

2.2 Discrete approximations

In order to conduct an empirical study, we need to convert the continuous-time model in Eqs. (2.1) and (2.3) into a discrete-time model. The discrete approximation of Eqs. (2.1) and (2.3) is as follows:

$$y_t=\psi\exp\left(\frac{h_t}{2}\right)u_t,\quad t=1,\ldots,n, \tag{2.4}$$

$$h_t=\beta h_{t-1}+\eta_t,\quad t=1,\ldots,n, \tag{2.5}$$

$$\begin{pmatrix}u_t\\ \eta_t\end{pmatrix}\sim i.i.d.\mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1&0\\0&\sigma_\eta^2\end{bmatrix}\right).$$

Here, y_t is the rate of return, $\psi^2\exp(h_t)=\sigma^2$ is the variance of rate of return y_t, and ψ is a scale parameter. u_t has mean 0 and variance 1 and η_t has mean 0 and variance σ_η^2, and these error terms each follow a normal distribution. Prefix "i.i.d." denotes independent and identically distributed. As shown, u_t and η_t are assumed to be uncorrelated or correlated. In the following, we simply refer to the discrete approximation SV model as the SV model.

3. SV model and Bayesian inference

3.1 SV model

The SV model, which is usually used in time series analysis in finance, describes the process of the rate of return y_t and volatility $\sigma_t^2 = \exp\ (h_t/2)$ as $h_t = \ln \sigma_t^2$, and is expressed as follows:

$$y_t = \exp\left(\frac{h_t}{2}\right)u_t, \quad t=1,\ldots,n, \tag{3.1}$$

$$h_{t+1} = \mu + \phi(h_t - \mu) + \eta_t, \quad t=0,\ldots,n-1, \tag{3.2}$$

$$\begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \sim i.i.d.\mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \sigma_\eta^2 \end{bmatrix}\right). \tag{3.3}$$

Here, h_t is a latent variable indicating the variation of volatility, μ is the mean volatility, and ϕ is a parameter indicating the persistence of shocks in volatility. For stationarity, we assume that $|\phi| < 1$. Equation (3.2) shows that the latent variable h_t follows a first-order autoregressive process.

Representing the unknown parameters (ϕ, σ_η, μ) collectively as $\boldsymbol{\theta}$, the likelihood function of the SV model can be expressed as follows:

$$\begin{aligned} L(\boldsymbol{\theta}) &= \int \cdots \int f(\boldsymbol{y}|\boldsymbol{h}) f(\boldsymbol{h}|\boldsymbol{\theta}) d\boldsymbol{h} \\ &= \int \cdots \int \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi \exp\ (h_t)}} \exp\left[-\frac{y_t^2}{2\exp\ (h_t)}\right] \\ &\times \prod_{t=1}^{T-1} \frac{1}{\sqrt{2\pi\sigma_\eta^2}} \exp\left[-\frac{\{h_{t+1} - \mu - \phi(h_t - \mu)\}^2}{2\sigma_\eta^2(1-\rho^2)}\right] \\ &\times \frac{\sqrt{1-\phi^2}}{\sqrt{2\pi\sigma_\eta^2}} \exp\left[-\frac{(1-\phi^2)(h_1-\mu)^2}{2\sigma_\eta^2}\right] dh_1 \ldots dh_T. \end{aligned} \tag{3.4}$$

Since this integral cannot be solved analytically, we propose using Bayesian estimation for parameter estimation.

3.2 Bayesian estimation

Let $\boldsymbol{\theta}$ be the unknown parameter set and $f(\boldsymbol{\theta})$ be the prior distribution of $\boldsymbol{\theta}$. In the Bayesian estimation method, we consider what kind of posterior distribution $f(\boldsymbol{\theta}|data)$ will replace prior distribution $f(\boldsymbol{\theta})$ before obtaining the data. Posterior distribution $f(\boldsymbol{\theta}|data)$ can be rewritten as follows from Bayes' theorem:

$$\begin{aligned} f(\boldsymbol{\theta}|data) &= \frac{f(\boldsymbol{\theta}, data)}{f(data)} \\ &= \frac{f(\boldsymbol{\theta}, data)}{\int f(\boldsymbol{\theta}, data) d\boldsymbol{\theta}} \\ &= \frac{f(data|\boldsymbol{\theta}) f(\boldsymbol{\theta})}{\int f(data|\boldsymbol{\theta}) f(\theta) d\boldsymbol{\theta}}. \end{aligned} \tag{3.5}$$

Here, $\int f(data\,|\,\boldsymbol{\theta})f(\boldsymbol{\theta})d\theta$ acts as a normalizing constant independent of $\boldsymbol{\theta}$ and can be ignored. Therefore, we can express the relationship as follows:

$$f(\boldsymbol{\theta}\,|\,data)\propto f(data\,|\,\boldsymbol{\theta})f(\boldsymbol{\theta}). \tag{3.6}$$

Here, $f(data\,|\,\boldsymbol{\theta})$ denotes the likelihood. In this case, the estimated value of parameter $\boldsymbol{\theta}$ is the expected value of the posterior （posterior mean） of the distribution. This can be expressed as follows:

$$E[f(\boldsymbol{\theta}\,|\,data)]=\int \boldsymbol{\theta}(\boldsymbol{\theta}\,|\,data)d\boldsymbol{\theta}. \tag{3.7}$$

In the case where the posterior distribution cannot be obtained analytically by Bayes' theorem, we can obtain the posterior distribution by sampling parameter $\boldsymbol{\theta}$ from posterior distribution $f(\boldsymbol{\theta}\,|\,data)$. In the case of nonlinear models, which are often used in economics and finance, the posterior distribution cannot be determined analytically in most cases.

The Gibbs sampling, the M-H, and the HMC methods are typically used in such cases. However, the conditional posterior distribution of the unknown parameters (ϕ,σ_η,μ) of the SV model requires the addition of the latent variable $\boldsymbol{h}$ to the condition, and it is important to know how to sample $\boldsymbol{h}$ efficiently. Therefore, we also treat the latent variable $\boldsymbol{h}$ as an unknown parameter. Although several methods that have been proposed so far – e.g., the single-move sampler, the multi-move sampler, and the mixture sampler – in this study, we propose using the HMC method.

3.3　Bayesian inference with HMC method

We would like to discuss Bayesian estimation using the HMC method in this section, but such content on the HMC method would overlap with that of Totsuka and Mitsui (2022), which reports other research of the same project. Therefore, for details on Bayesian estimation using the HMC method, we ask the reader to refer to Totsuka and Mitsui (2022), which uses basically the same algorithm for Bayesian estimation. For the calculation conditions and convergence diagnosis in Bayesian inference, we again ask the reader to refer to Totsuka and Mitsui (2022).

4. Extension of SV model

4.1　SV model with asymmetry

It is known that there is asymmetry in the relationship between the rate of return and the volatility in the stock market. In particular, when the rate of return falls, the volatility tends to rise in the next period, and when the rate of return rises, the volatility tends to fall in the next period (the leverage effect). This suggests that there is negative correlation between the rate of return and the volatility. To include such asymmetry in the model, we can consider adding the correlation between u_t and η_t to the model in Eqs. (3.1) and (3.2). If we construct the SV model assuming that u_t and η_t have correlation ρ, this is expressed as follows:

$$\begin{pmatrix} u_t \\ \eta_t \end{pmatrix}\sim i.i.d.\mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix},\begin{bmatrix} 1 & \rho\sigma_\eta \\ \rho\sigma_\eta & \sigma_\eta^2 \end{bmatrix}\right). \tag{4.1}$$

The model consisting of Eqs. (3.1), (3.2), and (4.1) is called the SVL model. When the unknown parameters $(\phi,\sigma_\eta,\rho,\mu)$ of the SVL model are collectively represented by $\boldsymbol{\theta}$, the likelihood function $f(\boldsymbol{y}\,|\,\boldsymbol{\theta})$ of the SVL model can be expressed as follows:

$$f(\boldsymbol{y}|\boldsymbol{\theta})=\int\cdots\int\prod_{t=1}^{n}\frac{1}{\sqrt{2\pi\exp(h_t)}}\exp\left[-\frac{y_t^2}{2\exp(h_t)}\right]$$
$$\times\prod_{t=1}^{n-1}\frac{1}{\sqrt{2\pi\sigma_\eta^2}\sqrt{1-\rho^2}}\exp\left[-\frac{\{h_{t+1}-\mu-\phi(h_t-\mu)-\rho\sigma_\eta y_t\exp(-h_t/2)\}^2}{2\sigma_\eta^2(1-\rho^2)}\right]$$
$$\times\frac{\sqrt{1-\phi^2}}{\sqrt{2\pi\sigma_\eta^2}}\exp\left[-\frac{(1-\phi^2)(h_1-\mu)^2}{2\sigma_\eta^2}\right]dh_1...dh_n. \tag{4.2}$$

Since this integral cannot be solved analytically, the parameters of the SVL model are difficult to estimate by maximum likelihood. Therefore, we propose using the HMC method along with the SV model.

4.2 SV model with fat-tails

The distribution of the rate of return on risky assets has long been known to be thicker than the normal distribution, as pointed out by Mandelbrot (1963) and Fama (1965). In order to consider an error distribution with a thick base, we introduce a random variable that follows the t-distribution. In this case, the process of the rate of return and volatility is described as follows:

$$y_t=\exp\left(\frac{h_t}{2}\right)\sqrt{z_t}u_t,\quad t=1,...,n, \tag{4.3}$$

$$h_{t+1}=\mu+\phi(h_t-\mu)+\eta_t,\quad t=0,...,n-1, \tag{4.4}$$

$$z_t\sim i.i.d.IG\left(\frac{\nu}{2},\frac{\nu}{2}\right). \tag{4.5}$$

where IG denotes the inverse gamma distribution, and the error term (u_t,η_t) follows Eq. (3.3) or (4.1). The $\sqrt{z_t}u_t$ term follows the t-distribution $z_t\sim i.i.d.IG(\nu/2,\nu/2)$, which has ν degrees of freedom. In this paper, the model consisting of Eqs. (3.3) and (4.3)-(4.5) is called the SVt model, and the model consisting of Eqs. (4.1) and (4.3)-(4.5) is called the SVLt model.

When the unknown parameters of the SVLt model $(\phi,\sigma_\eta,\rho,\mu,\nu)$ are collectively represented by $\boldsymbol{\theta}$, the likelihood function of the SVLt model can be expressed as follows:

$$L(\boldsymbol{\theta})=\int\cdots\int f(\boldsymbol{y}|\boldsymbol{h},\boldsymbol{z})f(\boldsymbol{h}|\boldsymbol{z},\boldsymbol{\theta})f(\boldsymbol{z}|\boldsymbol{\theta})d\boldsymbol{h}d\boldsymbol{z}$$
$$=\int\cdots\int\prod_{t=1}^{n}\frac{1}{\sqrt{2\pi z_t\exp(h_t)}}\exp\left[-\frac{y_t^2}{2z_t\exp(h_t)}\right]$$
$$\times\prod_{t=1}^{n-1}\frac{1}{\sqrt{2\pi\sigma_\eta^2}\sqrt{1-\rho^2}}\exp\left[-\frac{\{h_{t+1}-\mu-\phi(h_t-\mu)-\rho\sigma_\eta y_t\exp(-h_t/2)/\sqrt{z_t}\}^2}{2\sigma_\eta^2(1-\rho^2)}\right]$$
$$\times\frac{\sqrt{1-\phi^2}}{\sqrt{2\pi\sigma_\eta^2}}\exp\left[-\frac{(1-\phi^2)(h_1-\mu)^2}{2\sigma_\eta^2}\right]$$
$$\times\prod_{t=1}^{n}\frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)}z_t^{-(\nu/2+1)}\exp\left(-\frac{\nu}{2z_t}\right)dh_1...dh_ndz_1...dz_n. \tag{4.6}$$

Like in the SVL model case, we propose using Bayesian estimation to estimate the parameters of the SVLt model with the HMC method.

5. Application to Nikkei 225 Data

5.1 Data

In our empirical analysis, we used the closing prices of the Nikkei 225 obtained from Bloomberg. If S_t is the stock price at time t, then the daily rate of return y_t at time t is calculated as $y_t = (\ln S_t - \ln S_{t-1}) \times 100$. The data period of our dataset is January 5, 2015, to December 30, 2019, so the sample period for daily returns is January 6, 2015, to December 30, 2019, giving us a total of 1,221 observations. Table 1 shows the summary statistics for daily returns of the Nikkei 225 over our sample period. The negative skewness value indicates that the data follow a leftward skewed distribution. In addition, the fact that the excess kurtosis is above 0 indicates that the base of the distribution is thicker than that of a normal distribution.

Table 1: Summary statistics for Nikkei 225 daily returns

January 6, 2015 – December 30, 2019

No. of Obs.	Mean	Std. Dev.	Skewness	Exc. Kurtosis	Max.	Min.
1,221	0.0251	1.2257	−0.3414	6.1453	7.4262	−8.2529

5.2 Analytical model

The four SV models used in this study are summarized as follows:

1. SVn model: No leverage effect. The error follows a normal distribution.
2. SVLn model: The leverage effect is present (asymmetry is present). The error follows a normal distribution.
3. SVt model: No leverage effect. The error follows a t-distribution.
4. SVLt model: The leverage effect is present. The error follows a t-distribution.

The parameters of these models are estimated by Bayesian estimation using the HMC method.

5.3 Empirical results

Table 2 shows the estimation results of the SVn, SVLn, SVt, and SVLt models using the daily returns of the Nikkei 225 for each parameter in terms of the posterior mean and standard deviation, 95% credible interval, Gelman-Rubin statistic[6)], and inefficiency factor (IF).

The parameter ϕ expresses the persistence of shocks to volatility; as shown, the posterior mean of ϕ is around 0.83 in all models, indicating a high persistence of volatility. The parameter ρ represents the correlation between the daily rate of return and the volatility of Nikkei 225 and appears in the SVLn and SVLt models. Since the posterior distributions of ρ are −0.5308 and −0.5634, respectively, and furthermore, since the 95% credible intervals are [−0.6462, −0.4035] and [−0.6736, −0.4432], respectively, we can confidently conclude that ρ is negative. The posterior probability that ρ is negative is greater than 95% . This negative correlation between the daily rate of return of Nikkei 225 and the logarithm of volatility indicates the existence of the leverage effect.

The posterior mean of ν is 22.1928 in the SVt model and 20.3228 in the SVLt model, indicating that the distribution of the daily returns of the Nikkei 225 is thicker than a normal distribution. It is easily seen that the

Table 2: Estimation results for parameters

Line 1: Posterior mean (Std. Dev.), Line 2: 95% Credible interval,
Line 3: Gelman-Rubin statistics, Line 4: Inefficiency factor

	SVn	SVLn	SVt	SVLt
φ	0.8070 (0.0378)	0.8357 (0.0380)	0.8284 (0.0367)	0.8391 (0.0260)
	[0.7261, 0.8739]	[0.7631, 0.9126]	[0.7488, 0.8923]	[0.7837, 0.8854]
	1.0001	1.0001	1.0002	1.0002
	14.87	10.87	41.84	18.87
σ_η	0.6361 (0.0648)	0.5910 (0.0576)	0.5871 (0.0652)	0.5682 (0.0477)
	[0.5173, 0.7693]	[0.4901, 0.7009]	[0.4692, 0.7238]	[0.4792, 0.6672]
	1.0002	1.0002	1.0003	1.0002
	25.16	14.84	70.01	33.68
ρ		−0.5308 (0.0620)		−0.5634 (0.0592)
		[−0.6462, −0.4035]		[−0.6736, −0.4432]
		1.0002		1.0009
		7.33		17.25
ν			22.1928 (15.3968)	20.3228 (14.4584)
			[7.1048, 65.5541]	[7.2888, 59.5828]
			1.0032	1.0026
			257.67	178.83
μ	−0.1797 (0.1257)	−0.0702 (1.3422)	−0.2651 (0.1395)	−0.1813 (0.1200)
	[−0.4200, 0.0664]	[−0.3211, 0.1753]	[−0.5397, 0.0034]	[−0.4141, 0.0516]
	1.0000	1.0030	1.0001	1.0004
	3.05	4.96	32.23	26.59
ML	−2794.8802	−2507.5152	−3162.0423	−2942.1451

Gelman-Rubin statistic for each parameter is approximately 1 for all models; therefore, we can say that the obtained sample series converges well to the invariant distribution. Looking at marginal likelihood (ML)[7)], we see higher values for the t-distribution than for the normal distribution, and higher values without than with the leverage effect. Thus, among the four models, ML is highest for the SVt model, indicating that it is the best-fitting model for the daily returns of the Nikkei 225.

6. Conclusion

In this paper, we surveyed the Bayesian estimation of the SV model using the HMC method, and reported an empirical study using the daily returns data of Nikkei 225. As an extension of the SV model, we introduced an asymmetric SV model and an SV model with a fat-tailed distribution using the t-distribution. An empirical analysis revealed the existence of the leverage effect and the better fit of the model to the t-distribution than to the normal distribution. Furthermore, it was verified that the HMC method is effective in Bayesian estimation of SV models. Future work will include comparing the efficiency of the HMC method with other MCMC methods, such as the Gibbs sampling and the M-H methods.

Notes

1） For more information on SV models and their development, see Ghysels *et al.* (1996) and Shephard [ed.] (2005).

2） Empirical studies using Bayesian estimation of SV models with MCMC methods for returns on risky assets include Yu (2005), Omori *et al.* (2007), Omori and Watanabe (2008), Takahashi *et al.* (2009), and Nakajima and Omori (2010).

3） For details, see MacKay (2003, Chapter 30) and Neal (1994, 2011).

4） For details, see Alder and Wainwright (1956).

5） A model with asymmetry in volatility is sometimes referred to as an asymmetry SV model.

6） For details, see Gelman and Rubin (1992) and Gelman (1996).

7） For details, see Newton and Raftery (1994).

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