Empirical Study of Nikkei 225 Option with the Markov Switching GARCH Model *

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abstract

This paper estimated the price of Nikkei 225 Option with the Markov Switching GARCH Model, and evaluated the usefulness of this model in the option market. Assuming that investors are risk neutral, option prices were estimated through the Monte Carlo simulation. As a result of the empirical analysis, it turned out that it is extremely important, in the evaluation of option prices, to use the t-distribution for the distribution of underlying asset price return rate and adopt state variables that follow the Markov Switching process.

1 Introduction

The Black and Scholes (1973) model (hereinafter called "B-S model"), which is often used in the evaluation of European options [1)], assumes that volatility [2)] is constant until the maturity date. However, it is considered, from the results of many empirical analyses so far, that volatility changes over time, and so it is very important to formulate the variation in volatility and evaluate option prices. In order to understand the volatility variation clearly, Engle (1982) proposed the Autoregressive Conditional Heteroskedasticity (ARCH) model that formulates the volatility at each time as the linear function of the square of the past unexpected shock. In addition, Bollerslev (1986) added the past volatility values to the explanatory variables, and extended the GARCH (Generalized ARCH) model to a more general model

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1)The option that is exercisable only on the maturity date (right extinction date) is called a European option, and the option that is exercisable anytime until the maturity date is called an American option.

2)Volatility is defined based on the variance or standard deviation of the return on asset, and is used as the index of the risk of risky assets (assets whose return is uncertain, such as shares) in finance theory.

[3]. Empirical studies of options with such ARCH type models have been conducted by Engle and Mustafa (1992), Noh, Engle and Kane (1994), Saez (1997), Sabbatini and Linton (1998), Bauwens and Lubrano (1998), and Moriyasu (1999) [4]. In addition, empirical studies utilizing the GARCH model based on local risk neutrality, which was proposed by Duan (1995), have been conducted by Mitsui (2000), Duan and Zhang (2001), Bauwens and Lubrano (2002), Mitsui and Watanabe (2003), and Watanabe (2003). Incidentally, it is known that in the volatility variation model, including the ARCH model, the persistency of shock on volatility is extremely high. However, as Diabold (1986) and Lamoureux and Lastrapes (1990) pointed out, such persistency is considered to be caused by the structural change of volatility. Based on this fact, Hamilton and Susmel (1994) and Cai (1994) proposed the Morkov Switching ARCH (MS-ARCH) model by using a state variable that follows the Markov process in the formulation of the ARCH model, in order to take into account the structural change. Moreover, Gray (1996) proposed the Markov Switching GARCH (MS-GARCH) model by taking into account the structural change in the GARCH model, not the ARCH model. Satoyoshi (2004) conducted an empirical analysis of TOPIX (Tokyo Stock Exchange Price Index) with the MS-GARCH model, and found that the rate of TOPIX change underwent switching and that this model is superior to the conventional GARCH model in forecasting volatility of daily data. In addition, since the GARCH (1,1) model corresponds to the ARCH (∞) model, it is considered that the MS-GARCH model is more appropriate than the MS-ARCH model as a model describing the volatility variation used for the empirical analysis of option prices. In this study, we conducted an empirical study of option prices in the case where volatility follows the MS-GARCH model. The price of a European option like the Nikkei 225 Option can be obtained readily with the Monte Carlo simulation, by assuming the risk neutrality of investors. In addition, as a means for accelerating the convergence in the simulation, we adopted two variance reduction techniques: antithetic variates and control variates. The effectiveness of the MS-GARCH model in the Nikkei 225 Option market was studied by utilizing these techniques. The following 4 results were obtained from this empirical study. (1) When the MS-GARCH-t model is applied to a call option, the deviation rates of the estimated option price and the market price become the lowest. (2) When the GARCH-t model is applied to a put option, the deviation rates of the estimated option price and the market price become the lowest. (3) The option evaluation based on the MS-GARCH model, which was used in this study, can realize more appropriate pricing than the B-S model, which is

[3]With regard to the ARCH type model, refer to Bera and Higgins (1993) and Bollerslev, Engle and Nelson (1994), to review the statistical characteristics and methods, and Bollerslev, Chou and Kroner (1992) and Shephard (1996) to review the empirical study of finance.

[4]In these studies, the risk neutrality of investors was assumed, and so risk premium was not taken into account. Therefore, risk assets are evaluated based on only the expectation of the return on asset, and the expected return rate of risky assets becomes equal to that of risk-free assets.

the current standard in the option market. (4) The assumptions that underlying asset price return rate shows the t-distribution and that volatility follows the Markov Switching process are very important for evaluating option prices. The brief descriptions of the following chapters are as follows: Chapter 2 describes the formulation of profitability in the case where the MS-GARCH model and risk neutrality are assumed, and mentions a model for comparison in this study. Chapter 3 explains the method for evaluating a European Option by means of the Monte Carlo simulation. The results of the empirical analysis are summarized in Chapter 4. Chapter 5 contains conclusions and future study themes. Chapter 6 is a supplementary discussion.

2 Analytical Model

2.1 Markov Switching GARCH model

Gray (1996) proposed a model in which the parameters of the GARCH model depend on a state variable that follows the Markov process and undergoes switching. When the return rate at time t is defined as R_t and the underlying asset price at time t is defined as S_t, the underlying asset price return rate R_t at time t can be defined as follows:

$$R_t = \frac{S_t - S_{t-1}}{S_{t-1}} \ . \tag{2.1}$$

When volatility is represented by σ_t^2, the MS-GARCH model can be described as follows:

$$R_t = \mu + \epsilon_t, \tag{2.2}$$

$$\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d., E[z_t] = 0, Var[z_t] = 1, \tag{2.3}$$

$$\sigma_t^2 = \omega_{s_t} + \alpha_{s_t}\epsilon_{t-1}^2 + \beta_{s_t}E[\sigma_{t-1}^2|I_{t-2}], \tag{2.4}$$

$$\omega_{s_t} = \omega_0(1 - s_t) + \omega_1 s_t, \tag{2.5}$$

$$\alpha_{s_t} = \alpha_0(1 - s_t) + \alpha_1 s_t, \tag{2.6}$$

$$\beta_{s_t} = \beta_0(1 - s_t) + \beta_1 s_t. \tag{2.7}$$

The constant term μ in Equation (2.2) represents expected return rate, and ϵ_t depicts error term, and it is assumed that the return rate has no autocorrelation. $i.i.d.$ means "independent and identically distributed." $E[\cdot]$, $Var[\cdot]$, and $E[\cdot|\cdot]$ represent expectation, variance, and conditional expectation, respectively. Volatility σ_t^2 is the conditional variance of ϵ_t with the information set $I_{t-1} = \{R_{t-1}, R_{t-2}, \cdots\}$ up to time $t-1$ and the state variable s_t at time t being the conditions, that is, $\sigma_t^2 = Var[\epsilon_t|I_{t-1}, s_t]$. I_{t-2} in Equation (2.4) represents the information set $I_{t-2} = \{R_{t-2}, R_{t-3}, \cdots\}$ up to time $t-2$. s_t in Equations (2.5), (2.6), and (2.7) represents a state variable that follows the Markov process, and its transition probability is expressed as follows:

$$\Pr[s_t = 1|s_{t-1} = 1] = p, \quad \Pr[s_t = 0|s_{t-1} = 0] = q, \tag{2.8}$$

where $\Pr[s_t = j|s_{t-1} = i]$ is the probability of the transition from state i to state j.

Assuming that the volatility when $s_t = 0$ is σ_{0t}^2 and the volatility when $s_t = 1$ is σ_{1t}^2, volatility σ_t^2 becomes as follows:

$$\begin{cases} \sigma_{0t}^2 = \omega_0 + \alpha_0\epsilon_{t-1}^2 + \beta_0 E[\sigma_{t-1}^2|I_{t-2}], \text{ when } \ s_t = 0, \\ \sigma_{1t}^2 = \omega_1 + \alpha_1\epsilon_{t-1}^2 + \beta_1 E[\sigma_{t-1}^2|I_{t-2}], \text{ when } \ s_t = 1. \end{cases}$$

If the error term follows the normal distribution, z_t in Equation (2.3) becomes as follows:

$$z_t \sim i.i.d.N\left(0, 1\right). \tag{2.9}$$

If the error term follows the t-distribution, z_t becomes as follows:

$$z_t \sim i.i.d.t\left(0, 1, \nu\right). \tag{2.10}$$

Here, ν represents degree of freedom, and the variance of z_t has been standardized to be one.

Assuming that a state variable that follows the Markov Switching is directly introduced to the GARCH model, the equation for volatility becomes as follows:

$$\sigma_t^2 = \omega_{s_t} + \alpha_{s_t}\epsilon_{t-1}^2 + \beta_{s_t}\sigma_{t-1}^2.$$

However, in this model, σ_t^2 depends on not only s_t at time t but also all state variables $(s_t, s_{t-1}, \cdots, s_1)$ up to time t, and so it is impossible to conduct estimation with the maximum likelihood method. Accordingly, the model of Gray (1996) replaces the third term in the right-hand side σ_{t-1}^2 with $E[\sigma_{t-1}^2|I_{t-2}]$. Under the condition of I_{t-2}, σ_{t-1}^2 becomes equal to $\sigma_{0,t-1}^2$, when $\Pr[s_{t-1} = 0|I_{t-2}]$, and $\sigma_{1,t-1}^2$, when $\Pr[s_{t-1} = 1|I_{t-2}]$. Therefore, $E[\sigma_{t-1}^2|I_{t-2}]$ can be calculated as follows:

$$E[\sigma_{t-1}^2|I_{t-2}] = \sigma_{0,t-1}^2 \Pr[s_{t-1} = 0|I_{t-2}] + \sigma_{1,t-1}^2 \Pr[s_{t-1} = 1|I_{t-2}].$$

Here, σ_t^2 depends on only s_t at time t, and so it is possible to conduct estimation with the maximum likelihood method by obtaining the value of $Pr[s_t = j|I_{t-1}](j = 0, 1)$ by means of the filtering technique of Hamilton (1989) (Hamilton Filter). The detailed estimation method is described in the supplementary discussion in Section 6.1.

2.2 Risk Neutrality of Investors and Formulation of Return Rate

In this study, it is assumed that investors are risk neutral. Under this assumption, the expected rate of return μ becomes equal to the risk-free rate, and when the risk-free rate is represented by r, the underlying asset price return rate R_t in Equation (2.2) can be expressed as follows:

$$R_t = r + \epsilon_t. \tag{2.11}$$

Under the condition where the information I_{t-1} up to time $t-1$ is provided, the expectation in Equation (2.11) $E[R_t|I_{t-1}] = r$ is equal to r. R_t is expressed by Equation (2.1), and when it is substituted, the following equation is obtained:

$$E\left[\left.\frac{S_t - S_{t-1}}{S_{t-1}}\right| I_{t-1}\right] = r.$$

That is,

$$E[S_t|I_{t-1}] = S_{t-1}(1+r).$$

Then, risk neutrality can be confirmed.

Incidentally, in this section, the underlying asset price return rate is defined as Equation (2.1), but in theory of financial engineering, including options, R_t is expressed by the following equation with continuous compounding, in general:

$$R_t = \ln S_t - \ln S_{t-1}.$$

Here, it is assumed that the volatility σ_t^2 follows the ordinary GARCH model that does not include Markov Switching. Under the assumption of risk neutrality, the underlying asset price return rate can be formulated as follows:

$$R_t = r^* - \frac{1}{2}\sigma_t^2 + \epsilon_t, \tag{2.12}$$

where r^* is the interest rate of continuous compounding, and differs from r in Equation (2.11) [5]. Compared with Equation (2.11), it is obvious that the term $-(1/2)\,\sigma_t^2$ is added as the second term of the right-hand side in Equation (2.12). When z_t follows the standard normal distribution, the underlying asset price return rate follows a normal distribution with the following expectation and variance, under the condition that the information I_{t-1} up to time $t-1$ is provided,

$$E\left[R_t|I_{t-1}\right] = r^* - \frac{1}{2}\sigma_t^2, \quad Var\left[R_t|I_{t-1}\right] = \sigma_t^2.$$

When $\ln S_t$ is used, the following expression is obtained:

$$\ln S_t|I_{t-1} \sim N\left(\ln S_{t-1} + r^* - \frac{1}{2}\sigma_t^2, \sigma_t^2\right).$$

[5] Between r and the continuously-compounded interest rate r^*, there is the following relation: $r^* = \ln(1+r)$.

Therefore, it can be found that S_t follows a lognormal distribution having the following expectation:

$$\begin{aligned} E\left[S_t|I_{t-1}\right] &= \exp\left(\ln S_{t-1} + r^* - \frac{1}{2}\sigma_t^2 + \frac{1}{2}\sigma_t^2\right) \\ &= S_{t-1}\exp(r^*). \end{aligned}$$

This equation indicates that risk neutrality is true. Therefore, the formulation as Equation (2.12) can be made, when the underlying asset price return rate is calculated with the continuous compounding and z_t follows the standard normal distribution and volatility follows the normal GARCH model. However, if the error term z_t does not follow the normal distribution but follows the t-distribution, the second term in the right-hand side in Equation (2.12) must be modified, but it is impossible to obtain the new term analytically. Accordingly, in this study, the underlying asset price return rate is calculated as Equation (2.1) and Equation (2.11) is adopted.

2.3 Model for Comparison

This study analyzes the following normal GARCH model and the Markov Switching (MS) model, as well as the MS-GARCH model mentioned in Section 2.1.
GARCH model:

$$R_t = \mu + \epsilon_t,$$

$$\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d., E[z_t] = 0, Var[z_t] = 1, \tag{2.13}$$

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2. \tag{2.14}$$

MS model:

$$R_t = \mu + \epsilon_t,$$

$$\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d., E[z_t] = 0, Var[z_t] = 1, \tag{2.15}$$

$$\sigma_t^2 = \omega_0(1 - s_t) + \omega_1 s_t. \tag{2.16}$$

Here, the volatility σ_t^2 becomes either the ω_0 or ω_1 condition.

In this study, the following 6 kinds of models in which volatility changes and the B-S model are applied for the pricing of options, and these models are compared. "-n" implies that the error term follows the normal distribution, and "-t" means that the error term follows the t-distribution.

1. MS-GARCH-n $\cdots$ (2.3) – (2.9), (2.11).
2. MS-GARCH-t $\cdots$ (2.3) – (2.8), (2.10), (2.11).

3. GARCH-n $\cdots$ (2.9), (2.11), (2.13), (2.14).

4. GARCH-t $\cdots$ (2.10), (2.11), (2.13), (2.14).

5. MS-n $\cdots$ (2.9), (2.11), (2.15), (2.16).

6. MS-t $\cdots$ (2.10), (2.11), (2.15), (2.16).

7. B-S $\cdots$ B-S model (Black and Scholes (1973)).

Here, the European call option price C_T^{BS} and the European put option price P_T^{BS} at time T with an exercise price of K and a current maturity of τ can be obtained with the following B-S model.

$$C_T^{BS} = S_T N(d_1) - Ke^{-r^*\tau} N(d_2), \tag{2.17}$$

$$P_T^{BS} = -S_T N(-d_1) + Ke^{-r^*\tau} N(-d_2), \tag{2.18}$$

$$d_1 = \frac{\ln(S_T/K) + (r^* + \sigma^2/2)\tau}{\sigma\sqrt{\tau}},$$

$$d_2 = d_1 - \sigma\sqrt{\tau},$$

$$N(d_i) = \int_{-\infty}^{d_i} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx, \quad i = 1, 2.$$

Here, $N(\cdot)$ represents the distribution function of the standard normal distribution.

3 Method for Obtaining the Option Price

3.1 Option price under the assumption of risk neutrality

When investors are risk neutral, the price of a European Option becomes the present discounted value that is calculated by discounting the expectation of the option price at maturity with the interest rate of risk-free assets r. Namely, when it is assumed that the $T + \tau$ is maturity and that C_T is the price of the call option of the exercise price K at time T and that P_T is the put option price, the following expressions are obtained:

$$C_T = (1 + r)^{-\tau} E\left[Max\left(S_{T+\tau} - K, 0\right)\right], \tag{3.1}$$

$$P_T = (1 + r)^{-\tau} E\left[Max\left(K - S_{T+\tau}, 0\right)\right]. \tag{3.2}$$

Here, $S_{T+\tau}$ represents the underlying asset price at the maturity of the option. In the case of the MS-GARCH model, it is impossible to obtain the expectation in the right-hand side analytically, and so this is estimated by means of the Monte Carlo simulation [6)]. Simulation

[6)] As another method, Duan and Simonato (1998) proposed a method utilizing the empirical martingale simulation. The Monte Carlo experiment showed that the empirical martingale simulation is more efficient than the Monte Carlo simulation and the moment matching simulation developed by Barraquand (1995). In addition, Duan, Gauthier and Simonato (1999) concluded that the empirical martingale quasi-monte carlo simulation is more efficient than the empirical martingale simulation.

is conducted n times, to obtain n underlying asset prices at maturity $S_{T+\tau}$, and then these are expressed by $\left(S_{T+\tau}^{(1)}, S_{T+\tau}^{(2)}, \ldots, S_{T+\tau}^{(n)}\right)$, where $S_{T+\tau}^{(i)}$ represents the underlying asset price at maturity obtained through the i-th pass. When n is sufficiently large, the expectations in Equations (3.1) and (3.2) can be estimated with the following equations, because of the law of large numbers.

3.2 Procedures of the Monte Carlo Simulation

The procedures for calculating an option price with the Monte Carlo simulation in this study's model are as follows, where it is assumed that the error term of the MS-GARCH model follows the normal distribution.

[1] Estimate the unknown parameters of the MS-GARCH model with the maximum likelihood method, using the samples $\{R_1, R_2, \ldots, R_T\}$.

[2] Sample $\left\{z_{T+1}^{(i)}, z_{T+2}^{(i)}, \ldots, z_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ from independent standard normal distributions.

[3] Sample $\left\{u_{T+1}^{(i)}, u_{T+2}^{(i)}, \ldots, u_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ from independent standard rectangular distributions.

[4] Obtain the state variables following the Markov process $\left\{s_{T+1}^{(i)}, s_{T+2}^{(i)}, \ldots, s_{T+\tau}^{(i)}\right\}_{i=1}^{n}$, using uniform random numbers obtained at Step [3] and the transition probabilities p and q estimated with the maximum likelihood method.

[5] Calculate $\left\{R_{T+1}^{(i)}, R_{T+2}^{(i)}, \ldots, R_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ by substituting the values at Steps [2] and [4] into the MS-GARCH model.

[6] Obtain the underlying asset price $\left(S_{T+\tau}^{(1)}, S_{T+\tau}^{(2)}, \ldots, S_{T+\tau}^{(n)}\right)$ at the maturity time $T+\tau$ of the option with the following equation:

$$S_{T+\tau}^{(i)} = S_T \prod_{s=1}^{\tau} \left(1 + R_{T+s}^{(i)}\right), \quad i = 1, 2, \ldots, n. \tag{3.3}$$

[7] Calculate the call option's price C_T and the put option's price P_T with the following equation:

It is considered that the sufficient number of times of the Monte Carlo simulation is about 10,000 ($n = 10{,}000$). In order to reduce the variances of C_T and P_T, we propose the method of concurrently using the control variates and the antithetic variates, which are representative variance reduction techniques.

Incidentally, at Step [4], the state variables that follow the Markov process are obtained using uniform random numbers and transition probabilities, but this method cannot be applied for the state variable s_{T+1} at time $T+1$, the starting point. This is because even after the maximum likelihood method at Step [1], the value of the state variable s_T at time T remains unknown, and it is impossible to calculate the state variable s_{T+1} from uniform random numbers and transition probabilities. Accordingly, with regard to s_{T+1}, the following calculation is conducted utilizing the probability $\Pr[s_T = i|I_T]$ and the transition probability $\Pr[s_{T+1} = j|s_T = i]$.

$$\Pr[s_{T+1} = j|I_T] = \sum_{i=0}^{1} \Pr[s_{T+1} = j|s_T = i] \Pr[s_T = i|I_T].$$

From this probability, sampling is carried out.

4 Empirical Results of Nikkei 225 Option

4.1 Data

The options used for the empirical analysis in this study were Nikkei 225 call options (number of samples: 707) and put options (number of samples: 782) from May 2000 (expiration month) to MAR. 2006 (expiration month) [7)]. We analyzed the closing prices 20 business days ($\tau = 20$) before maturity of these options [8)]. As the data of the risk-free assets' interest rate r, the overnight unsecured call money was used [9)]. In addition, as the basic assumption, it was assumed that transaction costs, taxes, and dividends do not exist and any margin is not necessary for these options.

In order to estimate the parameters of the MS-GARCH model and the GARCH model, we used the closing prices of Nikkei 225 Stock Index 20 business days and 2,500 business days before maturity [10)]. For instance, in the case of the first expiration month, May 2000, the dates of option pricing are Apr. 11, 2000, which is 20 business days before maturity, and Feb. 21, 1990, which is 2,500 business days before maturity, and so when the daily change rate is calculated with Equation (2.1), the sampling period becomes from Feb. 22, 1990 to Apr. 11, 2000 (size of sample: T = 2,500). Using the daily return rates in this period, the model parameters are estimated, and based on the estimated parameters, the option prices are obtained through simulation. The same calculation is conducted from the next expiration month, and so 71 different sampling periods are defined for expiration months. In the case of the last expiration month, Mar. 2006, the day 20 business days before maturity is Feb.

[7)]The data of Nikkei 225 Option was provided by Osaka Securities Exchange.

[8)]The closing price of Nikkei 225 Option and the closing price of Nikkei 225 Stock Index may have been priced, but it was ignored in this study.

[9)]Nikkei NEEDS-FinancialQUEST was used as the data of the overnight unsecured call money.

[10)]Nikkei NEEDS-FinancialQUEST was used as the data of Nikkei 225 Stock Index (Nikkei Stock Average). In addition, the program language Ox (http://www.doornik.com/ox/) was used for estimating parameters.

Table 1: Summary Statistics for the Nikkei 225 Stock Index Daily Returns R_t

Sample Period: Feb. 22, 1990 – Feb. 10, 2006

Sample Size	Mean	Std Dev.	Skewness	Kurtosis	Max.	Min.	$LB^2(12)$
3935	-0.0001	0.0149	0.2898	6.4801	0.1324	-0.0698	231.0423
	(0.0002)		(0.0390)	(0.0781)			

note: ( ) denotes standard error. The standard error of the mean, skewness, and kurtosis estimates calculate $\hat{\sigma}/\sqrt{N}$, $\sqrt{6/N}$, and $\sqrt{24/N}$ respectively, where N=sample size and $\hat{\sigma}$ = standard deviation. $LB^2(12)$ is the heteroskedasticity-corrected Ljung = Box statistic following Diebold [1988].

10, 2006. Therefore, the entire sampling period of the daily change rates of Nikkei 225 Stock Index is from Feb. 22, 1990 to Feb. 10, 2006.

Table 1 shows the basic statistics of the daily change rates of Nikkei 225 Stock Index. The value of kurtosis is 6.4065, which is much larger than 3, the normal distribution's value, and so it is obvious that the distribution of the daily change rates has a thicker tail than a normal distribution. Such tail thickness may be caused by the temporal fluctuations in volatility. $LB^2(12)$ in the last line of the table is the Ljung-Box statistics for testing the null hypothesis that the first-order to twelfth-order autocorrelations when the daily change rate is squared are all zero [11)]. These statistics follow a χ^2 distribution with a degree of freedom of 12. The value of $LB^2(12)$ is 212.9053, which is very large. The critical value of the χ^2 distribution with a degree of freedom of 12 at the 1 % significance level is 26.22, and so it is considered that the daily change rate has significant nonlinear autocorrelation. From these results, it is found that it is necessary to use some volatility change model, like the model used in this study, in order to understand the temporal change of Nikkei 225 Stock Index.

Tables 2 to 7 tabulate the average, minimum, and maximum values of the parameters of each model in the 60 different sampling periods for each expiration month. From the results of the MS-GARCH-n model in Table 2, it is found that the averages of the transition probabilities are $p = 0.964$ and $q = 0.976$, which are very high. This indicates that once switching occurs, its state lingers for a long time. The persistency of the shock toward volatility in the GARCH part becomes as follows in each state: $\alpha_0+\beta_0 = 0.423$ and $\alpha_1+\beta_1 = 0.710$, and so it is obvious that the value of persistency varies. Such difference in persistency is also represented by the results of the MS-GARCH-t model in Table 3. However, persistency becomes higher on average in each state when it is assumed that the error term follows the t-distribution, compared with the case of a normal distribution.

According to the results of the GARCH-n model in Table 4 and the GARCH-t model in

[11)] Here, the heteroskedasticity of the Ljung-Box statistics has been modified by Diebold (1988).

Table 2: Estimation Results for MS-GARCH-n Model

$$R_t = r + \epsilon_t$$
$$\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d.N(0,1)$$
$$\sigma_t^2 = \omega_{s_t} + \alpha_{s_t}\epsilon_{t-1}^2 + \beta_{s_t}E[\sigma_{t-1}^2|I_{t-2}]$$
$$\omega_{s_t} = \omega_0(1-s_t) + \omega_1 s_t$$
$$\alpha_{s_t} = \alpha_0(1-s_t) + \alpha_1 s_t$$
$$\beta_{s_t} = \beta_0(1-s_t) + \beta_1 s_t$$

	p	q	ω_0	ω_1	α_0	α_1	β_0	β_1
Mean	0.925	0.963	0.439	1.464	0.001	0.019	0.452	0.811
Min.	0.514	0.822	0.000	0.000	0.000	0.000	0.321	0.527
Max.	0.979	0.984	0.634	2.526	0.014	0.078	0.674	2.048

	$\alpha_0+\beta_0$	$\alpha_1+\beta_1$	Log-likelihood
Mean	0.453	0.831	−4352.381
Min.	0.321	0.579	−4407.960
Max.	0.674	2.048	−4264.429

Table 5, the persistency of volatility is as follows: $\alpha+\beta = 0.973$ and $\alpha+\beta = 0.983$, which are nearly one. Such high persistency is the same as the results of most previous studies. Compared with the results of the MS-GARCH model, the value of the volatility's persistency is smaller in the MS-GARCH model than the GARCH model, regardless of whether the error term follows a normal distribution or the t-distribution. From this, it is found that the persistency of volatility described in the GARCH model decreases when the state variable that follow the Markov Switching is used in the GARCH model. In addition, when the average of the log likelihood of each model is compared, the highest value is in the MS-GARCH-t model, and the next is in the GARCH-t model. In order to judge which model is appropriate, it is necessary to test whether or not switching occurs. However, as commonly known, under the null hypothesis of no Markov Switching it is impossible to distinguish some parameters in the models, and the test statistics do not follow any ordinary asymptotic distribution; and so it becomes difficult to conduct the likelihood ratio test. Some test methods considering this problem were proposed by Hansen (1992, 1996) and Garcia (1998), but their methods were not conducted in this study because the objective of this study is to evaluate the option price.

Table 3: Estimation Results for MS-GARCH-t Model

$$
\begin{aligned}
&R_t = r + \epsilon_t \\
&\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d.t(0,1,\nu) \\
&\sigma_t^2 = \omega_{s_t} + \alpha_{s_t}\epsilon_{t-1}^2 + \beta_{s_t} E[\sigma_{t-1}^2 | I_{t-2}] \\
&\omega_{s_t} = \omega_0(1-s_t) + \omega_1 s_t \\
&\alpha_{s_t} = \alpha_0(1-s_t) + \alpha_1 s_t \\
&\beta_{s_t} = \beta_0(1-s_t) + \beta_1 s_t
\end{aligned}
$$

	p	q	ω_0	ω_1	α_0	α_1	β_0	β_1
Mean	0.994	0.991	0.206	0.333	0.013	0.065	0.605	0.872
Min.	0.986	0.985	0.012	0.089	0.000	0.046	0.103	0.790
Max.	1.000	1.000	0.696	0.735	0.066	0.097	0.904	0.902

	ν	$\alpha_0+\beta_0$	$\alpha_1+\beta_1$	Log-likelihood
Mean	8.145	0.618	0.937	−4321.839
Min.	6.930	0.103	0.875	−4382.226
Max.	9.929	0.970	0.965	−4249.719

Table 4: Estimation Results for GARCH-n Model

$$
\begin{aligned}
&R_t = r + \epsilon_t \\
&\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d.N(0,1) \\
&\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2
\end{aligned}
$$

	ω	α	β	$\alpha+\beta$	Log-likelihood
Mean	0.059	0.083	0.892	0.975	−4375.967
Min.	0.032	0.072	0.869	0.966	−4425.958
Max.	0.077	0.103	0.909	0.987	−4294.630

Table 5: Estimation Results for GARCH-t Model

$$
\begin{aligned}
&R_t = r + \epsilon_t \\
&\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d.t(0,1,\nu) \\
&\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2
\end{aligned}
$$

	ω	α	β	ν	$\alpha+\beta$	Log-likelihood
Mean	0.033	0.071	0.916	7.722	0.987	−4328.645
Min.	0.021	0.062	0.892	6.483	0.978	−4387.870
Max.	0.049	0.092	0.927	9.809	0.991	−4255.932

Table 6: Estimation Results for MS-n Model

$$R_t = r + \epsilon_t$$
$$\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d.N(0,1)$$
$$\sigma_t^2 = \omega_0(1 - s_t) + \omega_1 s_t$$

	p	q	ω_0	ω_1	Log-likelihood
Mean	0.962	0.983	1.191	4.408	−4366.123
Min.	0.947	0.980	1.110	3.983	−4421.448
Max.	0.970	0.987	1.291	5.249	−4294.827

Table 7: Estimation Results for MS-t Model

$$R_t = r + \epsilon_t$$
$$\epsilon_t = \sigma_t z_t, \quad \sigma_t > 0, \quad z_t \sim i.i.d.t(0,1,t)$$
$$\sigma_t^2 = \omega_0(1 - s_t) + \omega_1 s_t$$

	p	q	ω_0	ω_1	ν	Log-likelihood
Mean	0.976	0.988	1.226	4.076	10.322	−4352.254
Min.	0.968	0.985	0.980	3.051	7.802	−4410.443
Max.	0.990	0.990	1.340	4.976	14.889	−4279.640

Table 8: Moneyness

Moneyness	Call Option	Put Option
$S/K < 0.91$	deep-out-of-the-money (DOTM)	DITM
$0.91 \leq S/K < 0.97$	out-of-the-money (OTM)	ITM
$0.97 \leq S/K \leq 1.03$	at-the-money (ATM)	ATM
$1.03 < S/K \leq 1.09$	in-the-money (ITM)	OTM
$1.09 < S/K$	deep-in-the-money (DITM)	DOTM

4.2 Comparison of the Estimated Option Prices

Using the estimated values of option prices in the 7 kinds of models mentioned in Section 2.3 and actual market prices, the mean error rate (MER) and the root mean squared error rate (RMSER) are calculated, and each model is compared and discussed, as follows:

$$MER = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{\hat{X}_i^{\text{estimated}} - X_i^{\text{market price}}}{X_i^{\text{market price}}}\right), \tag{4.1}$$

$$RMSER = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(\frac{\hat{X}_i^{\text{estimated}} - X_i^{\text{market price}}}{X_i^{\text{market price}}}\right)^2}, \quad X = C, P. \tag{4.2}$$

Here, $\hat{X}_i^{\text{estimated}}$ is the option price estimated through the Monte Carlo simulation, or the theoretical price of the B-S model, and $X_i^{\text{market price}}$ represents the market option price. m is the number of samples. In addition, moneyness was categorized into the following 5 types by referring to the study of Bakshi, Cao and Chen (1997) (Refer to Table 8): (1) If $S/K < 0.91$, the call option is deep-out-of-the-money (DOTM) [12)], and the put option is deep-in-the-money (DITM) [13)]; (2) If $0.91 \leq S/K < 0.97$, the call option is out-of-the-money (OTM), and the put option is in-the-money (ITM); (3) If $0.97 \leq S/K \leq 1.03$, the call and put options are both at-the-money (ATM) [14)]; (4) If $1.03 < S/K \leq 1.09$, the call option is ITM, and the put option is OTM; (5) If $S/K > 1.09$, the call option is DITM, and the put option is DOTM. In the case of the call option, the number of samples is 216 for DOTM, 114 for OTM, 98 for ATM, 93 for ITM, and 186 for DITM. In the case of the put option, the number is 247 for DOTM, 96 for OTM, 98 for ATM, 99 for ITM, and 242 for DITM. The calculation results of MER and RMSER are tabulated in Tables 9 and 10.

12) It is also called "far-out-of-the-money."

13) It is also called "far-in-the-money."

14) Actually, it is rare that the option becomes ATM, and so the option around ATM is sometimes called "near-the-money" option.

Table 9: Estimation Results for Call Option

MER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
DOTM	0.2534	0.1980	0.1162	0.3893	0.4846	0.5281	−0.2174	216
OTM	0.3274	0.0436	0.2056	0.1465	0.4898	0.5115	−0.0977	114
ATM	0.0213	−0.0399	0.0094	−0.0149	0.0583	0.0693	−0.0883	98
ITM	−0.0083	−0.0145	−0.0120	−0.0134	−0.0051	−0.0010	−0.0211	93
DITM	−0.0044	−0.0034	−0.0042	−0.0032	−0.0041	−0.0030	−0.0023	186
Total	0.1309	0.0592	0.0673	0.1379	0.2334	0.2525	−0.0978	707

RMSER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
DOTM	1.2981	0.9976	1.0799	1.2861	1.8543	1.7660	1.3021	216
OTM	0.9948	0.4060	0.7144	0.5862	1.4988	1.4331	0.6333	114
ATM	0.2586	0.1417	0.2156	0.1868	0.3467	0.3366	0.2342	98
ITM	0.0803	0.0680	0.0755	0.0722	0.0842	0.0802	0.0830	93
DITM	0.0744	0.0743	0.0738	0.0736	0.0746	0.0745	0.0733	186
Total	0.8282	0.5792	0.6688	0.7535	1.1966	1.1410	0.7698	707

Table 10: Estimation Results for Put Option

MER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
DOTM	−0.5925	−0.6608	−0.7060	−0.6162	−0.5353	−0.5556	−0.8279	247
OTM	−0.0506	−0.2176	−0.1260	−0.1644	0.0285	0.0370	−0.3444	96
ATM	−0.0139	−0.0587	−0.0224	−0.0418	0.0166	0.0294	−0.0955	98
ITM	−0.0015	−0.0033	−0.0036	−0.0028	−0.0001	0.0055	−0.0025	99
DITM	−0.0001	−0.0003	−0.0007	0.0002	−0.0003	−0.0006	0.0025	242
Total	−0.1953	−0.2433	−0.2419	−0.2204	−0.1636	−0.1667	−0.3153	782

RMSER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
DOTM	0.7050	0.7263	0.7592	0.6832	0.7210	0.7282	0.8756	247
OTM	0.4175	0.3394	0.3294	0.3079	0.5987	0.5650	0.5116	96
ATM	0.2199	0.1548	0.1876	0.1722	0.2856	0.2788	0.2373	98
ITM	0.0918	0.0881	0.0870	0.0876	0.0933	0.0939	0.1014	99
DITM	0.0704	0.0709	0.0715	0.0717	0.0702	0.0703	0.0718	242
Total	0.4325	0.4316	0.4498	0.4066	0.4701	0.4680	0.5332	782

With regard to the call option, the following two points were clarified:

1. Under the MER standard, when the MS-GARCH-t model is used, the difference between the estimated option price and the market option price becomes the smallest, and it becomes the second smallest when the GARCH-n model is adopted. In addition, it was found that the model allowing volatility changes can correct the underpricing of the B-S model with respect to DOTM, OTM, ATM, and ITM. Especially, this is remarkable in the cases of DOTM and OTM.

2. Under the RMSER standard, when the MS-GARCH-t model is used, the rate of deviation between the estimated and market option prices becomes the smallest. This indicates that the performance of option pricing with the MS-GARCH-t model is the most outstanding, in the case of the call option.

In addition, the results for the put option are as follows:

1. Under the MER standard, when the MS-n model is used, the difference between the estimated and market option prices is the smallest.

2. Under the RMSER standard, when the GARCH-t model is used, the rate of deviation between the estimated and market option prices becomes the smallest, and it becomes the second smallest when the MS-GARCH-t model is adopted. This indicates that, in the case of the put option, the performance of option pricing with the GARCH-t model is the most outstanding.

In summary, it turned out that the option evaluation based on the MS-GARCH model used in this study can realize more appropriate pricing than the B-S model, which is now used as the bench mark in the option market. In addition, it was found that it is very important for the evaluation of option prices to assume that the underlying asset price return rate follows the t-distribution and that volatility undergoes the Markov Switching process.

4.3 Categorization based on Volume and Period until Maturity

The analyses so far were categorized based on moneyness, without taking into account the volume of options. However, in the actual option trading, there are a lot of options whose volume is extremely low, and there is a possibility that the price setting of such options is distorted. Accordingly, options were categorized into the 4 groups: the whole dealings, the dealings whose volume is over 50, the dealings whose volume is over 100, and the dealings whose volume is over 200. In addition, although the closing prices 20 business days ($\tau = 20$) before maturity of options have been analyzed, calculation is conducted in the cases of $\tau = 10$

and $\tau = 30$, too, because the option evaluation may vary according to the period until maturity.

The results for the call option are shown in Table 11. When seeing MER in the whole trading, the value of MER in the GARCH-t model is nearest to zero when $\tau = 10$, and the value of MER in the MS-GARCH-t model is nearest to zero when $\tau = 20$, the value of MER in the MS-GARCH-t model is nearest to zero when $\tau = 30$. This is the same even if the volume is limited to over 50, over 100, or over 200. With regard to RMSER in the whole dealings, the values in the MS-GARCH-t model become the smallest when $\tau = 10$ and $\tau = 20$, and the value in the GARCH-n model becomes the smallest when $\tau = 30$, and RMSER does not depend on volume. The results for the put option are shown in Table 12. The MER value becomes nearest to zero in the MS-t model regardless of volume and the period until maturity. In addition, the RMSER value becomes the smallest in the GARCH-t model, and RMSER does not depend on volume or the period until maturity. The above analysis results do not differ significantly from those mentioned in Section 4.2. Therefore, it became clear that it is still important to introduce a state variable that follows the Markov Switching to the volatility change and to assume the *t*-distribution for the distribution of the return rate of the underlying asset price, even if the options whose volume is low are removed or if the period until the maturity is altered.

Table 11: Results for Call Option: Categorization based on Volume and Period until Maturity

MER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
all								
$\tau = 10$	−0.0405	−0.0666	−0.0654	−0.0200	0.0552	0.0640	−0.1899	629
$\tau = 20$	0.1309	0.0592	0.0673	0.1379	0.2334	0.2525	−0.0978	707
$\tau = 30$	0.2823	0.1313	0.2360	0.2635	0.4008	0.4261	−0.0542	668
over 50								
$\tau = 10$	−0.0168	−0.0533	−0.0501	0.0104	0.1065	0.1258	−0.2340	403
$\tau = 20$	0.2073	0.0969	0.1135	0.2012	0.3424	0.3732	−0.1064	473
$\tau = 30$	0.4716	0.2145	0.3870	0.3763	0.6427	0.6640	−0.0751	397
over 100								
$\tau = 10$	−0.0067	−0.0596	−0.0426	0.0130	0.1196	0.1402	−0.2395	372
$\tau = 20$	0.2302	0.1069	0.1252	0.2140	0.3787	0.4103	−0.1026	423
$\tau = 30$	0.4913	0.2031	0.4035	0.3671	0.6962	0.7164	−0.0993	350
over 200								
$\tau = 10$	0.0106	−0.0468	−0.0255	0.0300	0.1396	0.1629	−0.2333	345
$\tau = 20$	0.2342	0.1161	0.1371	0.2196	0.3856	0.4213	−0.0870	393
$\tau = 30$	0.5444	0.2155	0.4334	0.3814	0.7703	0.7864	−0.1348	318

RMSER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
all								
$\tau = 10$	0.8128	0.6462	0.7837	0.6956	0.9727	0.9199	0.7027	629
$\tau = 20$	0.8282	0.5792	0.6688	0.7535	1.1966	1.1410	0.7698	707
$\tau = 30$	1.1683	0.7234	0.9351	0.8767	1.6490	1.5956	0.8231	668
over 50								
$\tau = 10$	0.9872	0.7726	0.9503	0.8388	1.1813	1.1208	0.8419	403
$\tau = 20$	0.9909	0.6764	0.7935	0.8900	1.4331	1.3715	0.9129	473
$\tau = 30$	1.4649	0.8845	1.1687	1.0375	2.0806	2.0007	0.9893	397
over 100								
$\tau = 10$	1.0184	0.7723	0.9785	0.8517	1.2206	1.1592	0.8530	372
$\tau = 20$	1.0191	0.6942	0.8112	0.8976	1.4910	1.4270	0.9571	423
$\tau = 30$	1.5016	0.8796	1.2006	1.0278	2.1972	2.1128	1.0179	350
over 200								
$\tau = 10$	1.0497	0.7934	1.0072	0.8776	1.2618	1.1976	0.8720	345
$\tau = 20$	1.0140	0.7135	0.8336	0.9219	1.4977	1.4409	0.9814	393
$\tau = 30$	1.5715	0.9151	1.2503	1.0631	2.3023	2.2129	1.0423	318

Table 12: Results for Put Option: Categorization based on Volume and Period until Maturity

MER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
all								
$\tau = 10$	-0.2255	-0.2372	-0.2524	-0.2249	-0.1923	-0.1831	-0.3046	696
$\tau = 20$	-0.1953	-0.2433	-0.2419	-0.2204	-0.1636	-0.1667	-0.3153	782
$\tau = 30$	-0.1328	-0.2188	-0.1692	-0.1551	-0.1086	-0.1075	-0.2965	718
over 50								
$\tau = 10$	-0.3002	-0.3221	-0.3462	-0.3039	-0.2432	-0.2287	-0.4363	403
$\tau = 20$	-0.2561	-0.3302	-0.3259	-0.2950	-0.2069	-0.2094	-0.4420	490
$\tau = 30$	-0.1769	-0.3316	-0.2497	-0.2355	-0.1331	-0.1367	-0.4744	412
over 100								
$\tau = 10$	-0.3161	-0.3399	-0.3658	-0.3204	-0.2544	-0.2388	-0.4634	371
$\tau = 20$	-0.2703	-0.3516	-0.3460	-0.3147	-0.2160	-0.2187	-0.4752	447
$\tau = 30$	-0.1799	-0.3505	-0.2610	-0.2582	-0.1249	-0.1300	-0.5034	375
over 200								
$\tau = 10$	-0.3211	-0.3475	-0.3742	-0.3271	-0.2554	-0.2394	-0.4788	345
$\tau = 20$	-0.2810	-0.3698	-0.3619	-0.3300	-0.2224	-0.2259	-0.5026	412
$\tau = 30$	-0.1690	-0.3547	-0.2557	-0.2568	-0.1067	-0.1124	-0.5213	327

RMSER

	MS-GARCH		GARCH		MS		B-S	m
	n	t	n	t	n	t		
all								
$\tau = 10$	0.4825	0.4549	0.4818	0.4468	0.4981	0.4860	0.5466	696
$\tau = 20$	0.4325	0.4316	0.4498	0.4066	0.4701	0.4680	0.5332	782
$\tau = 30$	0.4937	0.4122	0.4246	0.4078	0.5397	0.5145	0.5346	718
over 50								
$\tau = 10$	0.5621	0.5239	0.5605	0.5151	0.5856	0.5696	0.6539	403
$\tau = 20$	0.4923	0.4883	0.5096	0.4546	0.5458	0.5384	0.6236	490
$\tau = 30$	0.6117	0.5005	0.5202	0.5021	0.6752	0.6424	0.6686	412
over 100								
$\tau = 10$	0.5766	0.5362	0.5749	0.5279	0.6017	0.5850	0.6736	371
$\tau = 20$	0.5059	0.5019	0.5231	0.4680	0.5624	0.5537	0.6440	447
$\tau = 30$	0.6243	0.5157	0.5324	0.5020	0.6971	0.6631	0.6875	375
over 200								
$\tau = 10$	0.5823	0.5408	0.5794	0.5318	0.6100	0.5930	0.6835	345
$\tau = 20$	0.5189	0.5150	0.5360	0.4796	0.5794	0.5700	0.6618	412
$\tau = 30$	0.6416	0.5198	0.5379	0.5089	0.7203	0.6836	0.7037	327

5 Conclusion and Future Themes

In this paper, we focused on the option evaluation with the volatility-changing model, estimated the Nikkei 225 Option price with the GARCH model and its extended version-the MS-GARCH model proposed by Gray (1996)-and conducted an empirical test of the usefulness of the MS-GARCH model in the Nikkei 225 Option market. The major outcomes of this study are itemized below.

1. In the case of the call option, under the MER standard, the performance of option pricing with the MS-GARCH-t model is the most outstanding, and under the RMSER standard, the performance of option pricing with the MS-GARCH-t model is the most outstanding.

2. In the case of the put option, under the MER standard, the performance of option pricing with the MS-n model is the most outstanding, and under the RMSER standard, the performance of option pricing with the GARCH-t model is the most outstanding.

3. The option evaluation based on the MS-GARCH model, which was used in this study, can realize more appropriate pricing than the B-S model, which is now used as the standard in the option market.

4. It is very important in the evaluation of option prices to assume that the underlying asset price return rate follows the t-distribution and that volatility undergoes the Markov Switching process.

The future study subjects include the following four:

1. As the MS-GARCH model, Klaassen (2002) and Haas, Mittnik and Paolella (2004) also proposed models, and so it is necessary to make a comparison with the option evaluations based on these models

2. Make a comparison with the option prices and performance based on the stochastic volatility model, which is another representative volatility-changing model.

3. Conduct formulation, taking into account risk premium in the process of the underlying asset return rate, rather than assuming risk neutrality of investors.

4. Analyze the option prices in the volatility-changing model in detail. Study particularly implied volatility and volatility smile, etc.

6 Supplementary Discussion

6.1 Estimation Method based on the Maximum Likelihood Method in the MS-GARCH Model

The set of parameters is represented by θ. If the error term of the MS-GARCH model follows a normal distribution, $\theta = (\omega_0, \omega_1, \alpha_0, \alpha_1, \beta_0, \beta_1, p, q)$, and if the error term follows the t-distribution, a degree of freedom ν is added, that is, $\theta = (\omega_0, \omega_1, \alpha_0, \alpha_1, \beta_0, \beta_1, p, q, \nu)$. Then, the likelihood function $L(\theta)$ becomes as follows:

$$
\begin{aligned}
L(\theta) &= f(R_1, R_2, \cdots, R_T|\theta) \\
&= \prod_{t=1}^{T} f(R_t|I_{t-1};\theta) \\
&= \prod_{t=1}^{T} \sum_{s_t=0}^{1} f(R_t, s_t|I_{t-1};\theta) \\
&= \prod_{t=1}^{T} \sum_{s_t=0}^{1} f(R_t|s_t, I_{t-1};\theta) f(s_t|I_{t-1};\theta).
\end{aligned}
$$

The state variable s_t cannot be observed, and so the marginal density $f(R_t|I_{t-1};\theta)$ of R_t is obtained by adding the joint distributions $f(R_t, s_t|I_{t-1};\theta)$ of R_t and s_t in terms of s_t. The log-likelihood function becomes as follows:

$$
\ln L(\theta) = \sum_{t=1}^{T} \ln \left\{ \sum_{s_t=0}^{1} f(R_t|s_t, I_{t-1};\theta) f(s_t|I_{t-1};\theta) \right\}. \tag{6.1}
$$

If the error term z_t follows a normal distribution, the right-hand side $\{ \cdot \}$ of Equation (6.1) becomes as below,

$$
\begin{aligned}
\sum_{s_t=0}^{1} f(R_t|s_t, I_{t-1};\theta) f(s_t|I_{t-1};\theta) = &\frac{1}{\sqrt{2\pi\sigma_{0t}^2}} \exp\left(\frac{(R_t - r)^2}{2\sigma_{0t}^2}\right) \times \Pr[s_t = 0|I_{t-1}] \\
&+ \frac{1}{\sqrt{2\pi\sigma_{1t}^2}} \exp\left(\frac{(R_t - r)^2}{2\sigma_{1t}^2}\right) \times \Pr[s_t = 1|I_{t-1}],
\end{aligned} \tag{6.2}
$$

where the volatilities σ_{0t}^2 and σ_{1t}^2 are as follows:

$$
\begin{aligned}
\sigma_{0t}^2 &= \omega_0 + \alpha_0 \epsilon_{t-1}^2 + \beta_0 E\left[\sigma_{t-1}^2|I_{t-2}\right], \\
\sigma_{1t}^2 &= \omega_1 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 E\left[\sigma_{t-1}^2|I_{t-2}\right].
\end{aligned}
$$

$\Pr[s_t = 0|I_{t-1}]$ and $\Pr[s_t = 1|I_{t-1}]$ represents the probabilities of s_t when the information I_{t-1} until the time $t-1$ is provided. If the error term z_t follows the t-distribution, the right-hand side $\{\ \cdot\ \}$ becomes as follows:

$$
\begin{aligned}
&\sum_{s_t=0}^{1} f\left(R_t|s_t, I_{t-1};\theta\right) f\left(s_t|I_{t-1};\theta\right) \\
&\quad= \frac{\Gamma((\nu+1)/2)}{\pi^{\frac{1}{2}}\Gamma(\nu/2)}\left(1+\frac{(R_t-r)^2}{\sigma_{0t}^2(\nu-2)}\right)^{-\frac{\nu+1}{2}} \left(\sigma_{0t}^2\right)^{-\frac{1}{2}}(\nu-2)^{-\frac{1}{2}} \times \Pr\left[s_t=0|I_{t-1}\right] \\
&\quad+ \frac{\Gamma((\nu+1)/2)}{\pi^{\frac{1}{2}}\Gamma(\nu/2)}\left(1+\frac{(R_t-r)^2}{\sigma_{1t}^2(\nu-2)}\right)^{-\frac{\nu+1}{2}} \left(\sigma_{1t}^2\right)^{-\frac{1}{2}}(\nu-2)^{-\frac{1}{2}} \times \Pr\left[s_t=1|I_{t-1}\right]. \qquad (6.3)
\end{aligned}
$$

$\Pr\left[s_t=0|I_{t-1}\right]$ and $\Pr\left[s_t=1|I_{t-1}\right]$ in Equations (6.2) and (6.3) are obtained with the filtering method proposed by Hamilton (1989) (Hamilton Filter). In the following equations, $i=0,1$, $j=0,1$ represent the states at the time $t-1$ and at the time t, respectively. In order to obtain the probability of $s_t=j$ when the information I_t until the time t is provided, that is, $\Pr\left[s_t=j|I_t\right]$, $\Pr\left[s_{t-1}=i|I_{t-1}\right]$ is first calculated from $\Pr\left[s_t=j|I_{t-1}\right]$ with the following equation:

$$
\begin{aligned}
\Pr\left[s_t=j|I_{t-1}\right] &= \sum_{i=0}^{1} \Pr\left[s_t=j, s_{t-1}=i|I_{t-1}\right] \\
&= \sum_{i=0}^{1} \Pr\left[s_t=j|s_{t-1}=i\right] \Pr\left[s_{t-1}=i|I_{t-1}\right], \qquad (6.4)
\end{aligned}
$$

where $\Pr\left[s_t=j|s_{t-1}=i\right]$ is the transition probability calculated in Equation (2.8). Next, when the data R_t at the time t is added, the following equation is obtained:

$$
\begin{aligned}
\Pr\left[s_t=j|I_t\right] = \Pr\left[s_t=j|I_{t-1}, R_t\right] &= \frac{f\left(s_t=j, R_t|I_{t-1}\right)}{f\left(R_t|I_{t-1}\right)} \\
&= \frac{f\left(R_t|s_t=j, I_{t-1}\right)\Pr\left[s_t=j|I_{t-1}\right]}{\sum_{j=0}^{1} f\left(R_t|s_t=j, I_{t-1}\right)\Pr\left[s_t=j|I_{t-1}\right]}. \qquad (6.5)
\end{aligned}
$$

With this equation, $\Pr\left[s_t=j|I_t\right]$ is calculated, where $I_t = (I_{t-1}, R_t)$. By repeating the calculations of Equations (6.4) and (6.5), $\Pr\left[s_t=j|I_{t-1}\right]$ is obtained for $t=1,2,\ldots,T$, and the results are substituted into Equation (6.2) or (6.3). As $\Pr\left[s_0=i|I_0\right]$ which is necessary for the calculation of the time $t=1$, the following steady-state probability is used in general:

$$
\begin{aligned}
\pi_0 = \Pr\left[s_0=0|I_0\right] &= \frac{1-p}{2-p-q}, \\
\pi_1 = \Pr\left[s_0=1|I_0\right] &= \frac{1-q}{2-p-q}.
\end{aligned}
$$

6.2 Variance Reduction Techniques

This study proposed the use of the two variance reduction techniques: antithetic variates and control variates, in order to reduce the variance of the estimated values of the Monte

Carlo simulation and conduct more precise estimation [15)].

The antithetic variates is a method of producing two sequences that are negatively correlated with each other when random numbers are generated and calculating the average values of them to decrease the error in sampling. In the model of this paper, when $\left\{z_{T+1}^{(i)}, z_{T+2}^{(i)}, \ldots, z_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ is sampled from the standard normal distribution at Step [2], minus is added to it to obtain $\left\{-z_{T+1}^{(i)}, -z_{T+2}^{(i)}, \ldots, -z_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ and add it to random numbers. At Step [3], $\left\{u_{T+1}^{(i)}, u_{T+2}^{(i)}, \ldots, u_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ is sampled from the standard uniform distribution, in the same way, and then the value subtracting the uniform random number from one: $\left\{1-u_{T+1}^{(i)}, 1-u_{T+2}^{(i)}, \ldots, 1-u_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ is added. Therefore, the number of times of the Monte Carlo simulation after Step [4] is $2n$. There emerges a negative correlation between the underlying asset prices at maturity calculated using the above two kinds of random number sequences—$\left\{S_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ and $\left\{S_{T+\tau}^{(i)}\right\}_{i=n+1}^{2n}$, and so it is possible to reduce the variance of the option prices.

The control variates, the other variance reduction technique, is the method of defining the analytically-calculable variables as control variables and reducing variance by using the analytically calculated control variables and the values obtained through the Monte Carlo simulation. As the control variable of the control variates, the option price of the B-S model is used. In the B-S model, it is assumed that the underlying asset price S follows the geometric Brownian motion.

$$dS = \mu S dt + \sigma S dW.$$

where μ is the expected return rate, dt is the infinitesimal time interval, σ is the standard deviation, and dW is the infinitesimal increase in the standard Wiener process. At this time, the natural logarithm of the underlying asset price $\ln S$ is expressed by the following equation based on the Ito's formula:

$$d \ln S = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW.$$

$\ln S$ follows the arithmetic Brownian motion. Here, when S_T represents the underlying asset price at the time T, which is the time for evaluating the option price, and $S_{T+\tau}$ depicts the underlying asset price at the time $T+\tau$, the difference between each natural logarithm $\ln S_{T+\tau} - \ln S_T$ follows the following normal distribution:

$$\ln S_{T+\tau} - \ln S_T \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right).$$

[15)] In addition to these techniques, a variety of techniques has been proposed, including the stratified sampling, the Latin hypercube sampling, and the importance sampling. With regard to the option evaluation based on numerical calculation, refer to Broadie and Glasserman (1996), Boyle, Broadie and Glasserman (1997), Ross (2002, Chapter 8), Jäckel (2002), Seydel (2002, Chapter 3), and Tavella (2002, Chapter 5, 6), etc.

In this paper, the risk neutrality of investors is assumed, and so μ becomes equal to the risk-free assets' continuously-compounded interest rate r^*. Therefore, the underlying asset price return rate $R_t = \ln S_t - \ln S_{t-1}$ at the time t can be formulated as follows:

$$\begin{aligned} R_t &= r^* - \frac{1}{2}\sigma^2 + \epsilon_t, \\ \epsilon_t &= \sigma z_t, \quad z_t \sim i.i.d.N\left(0,1\right). \end{aligned} \tag{6.6}$$

In addition, it is possible to rewrite $\ln S_{T+\tau} - \ln S_T$ as follows:

$$\begin{aligned} &\ln S_{T+\tau} - \ln S_T \\ &\quad = (\ln S_{T+\tau} - \ln S_{T+\tau-1}) + (\ln S_{T+\tau-1} - \ln S_{T+\tau-2}) + \cdots + (\ln S_{T+1} - \ln S_T) \\ &\quad = R_{T+\tau} + R_{T+\tau-1} + \cdots + R_{T+1}. \end{aligned} \tag{6.7}$$

Then, the underlying asset price at maturity $S_{T+\tau}^{(i)}$, which is obtained through the i-th pass, can be expressed by the following equation:

$$\begin{aligned} S_{T+\tau}^{(i)} &= S_T \exp\left(R_{T+\tau}^{(i)} + R_{T+\tau-1}^{(i)} + \cdots + R_{T+1}^{(i)}\right) \\ &= S_T \exp\left\{\left(r^* - \frac{1}{2}\sigma^2 + \epsilon_{T+\tau}^{(i)}\right) + \left(r^* - \frac{1}{2}\sigma^2 + \epsilon_{T+\tau-1}^{(i)}\right) + \cdots + \left(r^* - \frac{1}{2}\sigma^2 + \epsilon_{T+1}^{(i)}\right)\right\} \\ &= S_T \exp\left(r^*\tau - \frac{1}{2}\sigma^2\tau + \sigma \sum_{t=T+1}^{T+\tau} z_t^{(i)}\right), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{6.8}$$

Therefore, this equation is used for obtaining the underlying asset price at maturity with the Monte Carlo simulation based on the B-S model. In general, Historical Volatility (HV) is used for the standard deviation σ. Historical Volatility means the volatility calculated from past stock data. In this study, the standard deviation of the underlying asset price change rate in the past 20 days is used, and HV is calculated as follows:

$$\sigma_{HV} = \sqrt{\frac{1}{20-1}\sum_{t=1}^{20}(R_t - \bar{R})^2}. \tag{6.9}$$

where $\bar{R}$ is the mean of R_t in 20 days.

When $S_{MS-GARCH}^{(i)}$ represents the underlying asset price at the maturity $T+\tau$, which is calculated through the Monte Carlo simulation based on the MS-GARCH model, $S_{BS}^{(i)}$ represents the underlying asset price at maturity, which is calculated through the Monte Carlo simulation based on the B-S model, $\tilde{C}_{MS-GARCH}$, $\tilde{C}_{BS}$ and C_{BS} represent the call option prices at the time T in respective models, and C_{BS} depicts the analytic solution in the B-S model [16] , the call option price can be calculated as follows:

$$C_T = \tilde{C}_{MS-GARCH} - \varphi\left(\tilde{C}_{BS} - C_{BS}\right). \tag{6.10}$$

The expectations of both sides of the above equation are calculated as follows:

$$\begin{aligned} E\left[C_T\right] &= E\left[\tilde{C}_{MS-GARCH} - \varphi\left(\tilde{C}_{BS} - C_{BS}\right)\right] \\ &= E\left[\tilde{C}_{MS-GARCH}\right] - \varphi\left(C_{BS} - C_{BS}\right) \\ &= E\left[\tilde{C}_{MS-GARCH}\right]. \end{aligned}$$

It is obvious that the expectation of C_T in the left-hand side, which is obtained through the Monte Carlo simulation, is equal to the expectation of $\tilde{C}_{MS-GARCH}$, which is calculated through the Monte Carlo simulation based on the MS-GARCH model. In addition, using Equation (6.10), the variance of C_T is expressed by the following equation:

$$Var\left(C_T\right) = Var\left(\tilde{C}_{MS-GARCH}\right) + \varphi^2 Var\left(\tilde{C}_{BS}\right) - 2\varphi Cov\left(\tilde{C}_{MS-GARCH}, \tilde{C}_{BS}\right).$$

φ is obtained by partially differentiating the above equation in terms of φ, which minimizes the variance.

$$\varphi = \frac{Cov\left(\tilde{C}_{MS-GARCH}, \tilde{C}_{BS}\right)}{Var\left(\tilde{C}_{BS}\right)}. \tag{6.11}$$

The calculation for the put option is conducted in the same way.

If the error term of the MS-GARCH model follows the t-distribution as expressed in Equation (2.10), $\left\{z_{T+1}^{(i)}, z_{T+2}^{(i)}, \ldots, z_{T+\tau}^{(i)}\right\}_{i=1}^{n}$ is sampled from the t-distribution with a degree of freedom of ν and a variance of 1, not the standard normal distribution, at Step [2]. To conduct this sampling, $x_t^{(i)}$ and $w_t^{(i)}$ are first sampled from the standard normal distribution and the χ^2 distribution with a degree of freedom of ν, which are independent of each other, and then the following calculation is carried out.

$$z_t^{(i)} = \frac{\sqrt{\nu - 2}\, x_t^{(i)}}{\sqrt{w_t^{(i)}}}.$$

In this case, when the option price of the B-S model is obtained through the Monte Carlo simulation by means of the control variates, the calculation is conducted using $x_t^{(i)}$ instead of $z_t^{(i)}$ in Equation (6.8).

[16] The Historical Volatility of Equation (6.9) is also used for the volatility σ of the B-S model, so that it becomes consistent with the B-S solution based on simulation. Here, since the B-S model needs the annualized volatility, the number of trading days in a year is 250, and so the volatility σ of the B-S model is multiplied by $\sqrt{250}$.

References

[1] Bakshi, G., C. Cao and Z. Chen (1997), "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 52, pp.2003–2049.

[2] Barraquand, J. (1995), "Numerical Valuation of High Dimentional Multivariate European Securities," *Management Science*, 41, pp.1882–1891.

[3] Bauwens, L. and M. Lubrano (1998), "Bayesian Inference on GARCH Models Using the Gibbs Sampler," *Econometrics Journal*, 1, pp.C23–C46.

[4] Bauwens, L. and M. Lubrano (2002), "Bayesian Option Pricing Using Asymmetric GARCH Models," *Journal of Empirical Finance*, 9, pp.321–342.

[5] Bera, A. K. and M. L. Higgins (1993), "On ARCH Models: Properties, Estimation and Testing," *Journal of Economic Surveys*, 7, pp.305–366.

[6] Black, F. and M. Scholes (1973), "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, pp.637–654.

[7] Bollerslev, T. (1986), "Generalized Autoregressive Conditional Heteroskedasticity," *Journal of Econometrics*, 31, pp.307–327.

[8] Bollerslev, T., R. Y. Chou and K. F. Kroner (1992), "ARCH Modeling in Finance: A Review of the Theory and Empirical Evidence," *Journal of Econometrics*, 52, pp.5–59.

[9] Bollerslev, T., R. F. Engle and D. B. Nelson (1994), "ARCH Models," R. F. Engle and D. McFadden (eds.), *The Handbook of Econometrics*, Vol.4, pp.2959–3038, North-Holland.

[10] Boyle, P., M. Broadie and P. Glasserman (1997), "Monte Carlo Methods for Security Pricing," *Journal of Economic Dynamics and Control*, 21, pp.1267–1321.

[11] Broadie, M. and P. Glasserman (1996), "Estimating Security Price Derivatives Using Simulation," *Management Science*, 42, pp.269–285.

[12] Cai, J. (1994), "A Markov Model of Switching-Regime ARCH," *Journal of Business & Economic Statistics*, 12, pp.309–316.

[13] Diebold, F. X. (1986), "Modeling the Persistence of Conditional Variances: A Comment," *Econometric Reviews*, 5, pp.51–56.

[14] Diebold, F. X. (1988). *Empirical Modeling of Exchange Rate Dynamics*, Springer-Verlag.

[15] Duan, J.-C. (1995), "The GARCH Option Pricing Model," *Mathematical Finance*, 5, pp.13–32.

[16] Duan, J. -C. and J. -G. Simonato (1998), "Empirical Martingale Simulation for Asset Prices," *Management Science*, 44, pp.1218–1233.

[17] Duan, J. -C. and H. Zhang (2001), "Pricing Hang Seng Index Options around the Asian Financial Crisis - A GARCH Approach," *Journal of Banking & Finance*, 25, pp.1989–2014.

[18] Duan, J. -C., G. Gauthier and J. -G. Simonato (1999), "An Analytical Approximation for the GARCH Option Pricing Model," *Journal of Computational Finance*, 2, pp.75–116.

[19] Engle, R. F. (1982), "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation," *Econometrica*, 50, pp.987–1007.

[20] Engle, R. F. and C. Mustafa (1992), "Implied ARCH Models from Options Prices," *Journal of Econometrics*, 52, pp.289–311.

[21] Garcia, R. (1998), "Asymptotic Null Distribution of the Likelihood Ratio Test in Markov Switching Models," *International Economic Review*, 39, pp.763-788.

[22] Gray, S. F. (1996), "Modeling the Conditional Distribution of Interest Rates as a Regime-Switching Process," *Journal of Financial Economics*, 42, pp.27–62.

[23] Haas, M., S. Mittnik and M. S. Paolella (2004), "A New Approach to Markov-Switching GARCH Models," *Journal of Financial Econometrics*, 2, pp.493–530.

[24] Hamilton, J. D. (1989), "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle," *Econometrica*, 57, pp.357–384.

[25] Hamilton, J. D. and R. Susmel (1994), "Autoregressive Conditional Heteroskedasticity and Changes in Regime," *Journal of Econometrics*, 64, pp.307–333.

[26] Hansen, B. E. (1992), "The Likelihood Ratio Test under Nonstandard Conditions: Testing the Markov Switching Model of GNP," *Journal of Applied Econometrics*, 7, S61-S82.

[27] Hansen, B. E. (1996), "Erratum: The Likelihood Ratio Test under Nonstandard Conditions: Testing the Markov Switching Model of GNP," *Journal of Applied Econometrics*, 11, pp.195-198.

[28] Jäckel, P. (2002), *Monte Carlo Methods in Finance*, John Wiley & Sons.

[29] Klaassen, F. (2002), "Improving GARCH Volatility Forecasts with Regime-Switching GARCH," *Empirical Economics*, 27, pp.363–394.

[30] Lamoureux, C. G. and W. D. Lastrapes (1990), "Persistence in Variance, Structural Change, and the GARCH Model," *Journal of Business & Economic Statistics*, 8, pp.225–234.

[31] Mitsui, H. (2000), "An Analysis of Nikkei 225 Option Pricing by a GARCH Model," *Gendai-Finance*, 7, pp.57–73 (in Japanese).

[32] Mitsui, H. and T. Watanabe (2003), "Bayesian Analysis of GARCH Option Pricing Models," *Journal of the Japan Statistical Society*, 33, pp.307–324 (in Japanese).

[33] Moriyasu, H. (1999), "Valuing the Nikkei 225 Option Using the ARCH Option Pricing Method," *Studies in Contemporary Economics*, 7, pp.143–159 (in Japanese).

[34] Noh, J., R. F. Engle and A. Kane (1994), "Forecasting Volatility and Option Pricing of the S&P500 Index," *Journal of Derivatives*, pp.17–30, Reprinted in Engle, R. F. (ed.), *ARCH: Selected Readings*, (1995), Chapter 15, Oxford University Press.

[35] Ross, S. M. (2002), *Simulation*, 3rd ed., Academic Press.

[36] Sabbatini, M. and O. Linton (1998), "A GARCH Model of the Implied Volatility of the Swiss Market Index from Option Prices," *International Journal of Forecasting*, 14, pp.199–213.

[37] Saez, M. (1997), "Option Pricing under Stochastic Volatility and Interest Rate in the Spanish Case," *Applied Financial Economics*, 7, pp.379–394.

[38] Satoyoshi, K. (2004) "An Empirical Study of Volatility in the Japanese Stock Market Using the Markov Switching GARCH Model," *Journal of the Japan Statistical Society*, 34, pp.1–19 (in Japanese).

[39] Shephard, N. (1996), "Statistical Aspects of ARCH and Stochastic Volatility," in D. R. Cox, D. V. Hinkley and O. E. Barndorff-Nielsen (eds.), *Time Series Models in Econometrics, Finance and other Fields*, No.65 in Monographs on Statistics and Applied Probability, pp.1–67, Chapman & Hall.

[40] Seydel, R. (2002), *Tools for Computational Finance*, Springer-Verlag.

[41] Tavella, D. (2002), *Quantitative Methods in Derivatives Pricing*, John Wiley & Sons.

[42] Watanabe, T. (2003), "An Analysis of the GARCH Option Pricing Model Using the Nikkei 225 Option Data," *Monetary and Economics Studies*, 22, pp.1–34 (in Japanese).