Environmental Pollution in a Delay Solow Model with Constant Population Growth[*]

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1 Introduction

The analysis of economic dynamics has a long history, and scholars have developed various methods to study the impack of delays. Kalecki's (1935) pioneering work on delayed macroeconomic dynamics is notable because the Kaleckian model exhibits persistent oscillations when a production delays are taken into account. After more than fifty years, Kaleckian delays were revisited in a Kaldor-Kaldor model by Krawiec and Szydłowski (1999) and in a Solow model by Zak (1999). These continuous-time models show that production delays can cause cyclic oscillations. Conversely, Day (1982) reexamined the dynamics of the Solow model in a discrete-time setting and showed that a Solow model with a strong pollution effect can produce various dynamics, ranging from periodic oscillations to chaos. More recently, Matsumoto and Szidarovszky (2011, 2013) reconstructed Day's difference model as a delay differential equation model, demonstrating its ability to generate both simple and complicated dynamics. In a recent study, Guerrini et al. (2019) examined the stabilizing and destabilizing effects of delays in a Solow model with a Cobb-Douglas production function. Matsumoto and Szidarovszky (2023) reexamined the various delay effects of a Solow model with a CES production function.

Building upon the theoretical and qualitative framework established in Guerrini et al. (2019), this paper explores the link between long-term economic growth and environmental pollution. It is crucial to examine an economic framework in which environmental degradation resulting from the use of capital in production processes has the potential to negatively affect productivity in the context of capital accumulation. Building on the analysis initially requested by Xepapades (2005), this study develops a model assuming that pollution is an inevitable byproduct of production, unifying the process of economic growth with the environmental dynamics. Ferrara et al. (2014) then introduced a time-to-build delay, also known as a production delay, providing a more detailed analysis. Specifically, they demonstrate analytically the emergence of a Hopf bifurcation when the delay passes through a threshold value. The present study aims to extend this line of research by focusing on the interdependency between capital and pollution dynamics.

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The paper is structured as follows. Section 2 incorporates the time-to-build delay into the production process, and reformulates the capital and pollution accumulation dynamics to account for the delay effects. The dynamic system is first summarized in the absengce of delay, and locally asymptotical stability is confirmed. Section 3 then shifts the focus to a simplified model that is devoid of factors influencing capital accumulation. It has been demonstrated that a threshold delay value exists, and that the stationary point is locally asymptotically stable when the delay is less than the threshold value. When the delay exceeds the threshold value, the stationary point bifurcates to a limit cycle. Section 4 shifts the focus back to the comprehensive model, examining the impact of pollution contamination on capital accumulation. Finally, Section 5 summarizes the key points and suggests topics for future research. The Appendix contains the complete derivations of the transversality crossing conditions.

2 Delay Solow Model

2.1 Basic Delay Framework

We develop a model describing the process of economic growth process in which environmental pollution is a byproduct of production. The aggregate production function for the economy is Cobb-Douglas,[1)]

$$Y(t) = [K(t-\tau)]^{\alpha} [A(t)L(t)]^{1-\alpha},\ 0 < \alpha < 1 \tag{1}$$

where $\tau > 0$ denotes a time delay and $A(t)L(t)$ is effective labor. $A(t)$ represents labor-augmenting technical change, however, it is assumed to be constant (i.e., $\dot{A}(t) = 0$ and $A(t) = 1$ for all $t \geq 0$). Capital accumulation is

$$\dot{K}(t) = s [K(t-\tau)]^{\alpha} [L(t)]^{1-\alpha} - \delta K(t-\tau) - \theta P(t). \tag{2}$$

Here, s is the constant marginal propensity to save ($0 < s < 1$), $\delta > 0$ is the constant depreciation rate of the capital stock. $\theta P(t)$ denotes decontamination activities and is carried out in proportion to the stock pollution where $\theta > 0$ is the detrimental pollution effect on the capital accumulation.[2)]

The population of the economy grows at a constant, nonzero rate $n \neq 0$,

$$\dot{L}(t) = nL(t). \tag{3}$$

It is assumed that one unit of production emits one unit of pollution,

$$P(t) = \phi Y(t) \tag{4}$$

where ϕ is the emissions per unit of output and is assumed to be constant.[3)] Pollution accumulates in the ambient environment according to

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[1)] The assumption of a Cobb-Douglas type production function is a matter of analytical convenience, and the same results can be also obtained under a standard constant returns to scale production function.

[2)] We can explain this formulation by assuming that pollution has a negative linear externality on the production process represented as $Y(t) = [K(t)]^{\alpha} [L(t)]^{1-\alpha} - \theta P(t)$ as discussed bySchou (2000) and Smulders and Gradus (1996). The resultant capital accumulation equation is

$$\dot{K}(t) = s [K(t)]^{\alpha} [L(t)]^{1-\alpha} - \delta K(t-\tau) - s\theta P(t)$$

This is essentially the same as equation (2).

[3)] $\phi = 1 - \eta$, η is an abatement parameter. Efficient abatement technology has a larger η and thus a smaller ϕ.

$$\dot{P}(t) = \phi \left[K(t-\tau)\right]^{\alpha} \left[L(t)\right]^{1-\alpha} - mP(t) \tag{5}$$

where $m > 0$ reflects exponential pollution decay.

Due to the linear homogenous production function, we can transform the dynamic equations in the extensive form to those in the intensive form.[4] Dividing the both sides of (2) by $L(t)$ yields

$$\frac{\dot{K}(t)}{L(t)} = s\left[\frac{K(t-\tau)}{L(t-\tau)}\frac{L(t-\tau)}{L(t)}\right]^{\alpha} - \delta\frac{K(t-\tau)}{L(t-\tau)}\frac{L(t-\tau)}{L(t)} - \theta\frac{P(t)}{L(t)}.$$

This expression can be rewritten as

$$\dot{k}(t) = s\left(\Delta_{\tau}\right)^{\alpha}\left[k(t-\tau)\right]^{\alpha} - \delta\left(\Delta_{\tau}\right)^{\alpha} k(t-\tau) - nk(t) - \theta p(t) \tag{6}$$

where

$$k(t) = \frac{K(t)}{L(t)},\ p(t) = \frac{P(t)}{L(t)},\ \Delta_{\tau} = \frac{L(t-\tau)}{L(t)} = e^{-n\tau}.$$

Similarly, dividing the both sides of (5) by $L(t)$ presents

$$\frac{\dot{P}(t)}{L(t)} = \phi\left[\frac{K(t-\tau)}{L(t-\tau)}\frac{L(t-\tau)}{L(t)}\right]^{\alpha} - m\frac{P(t)}{L(t)}.$$

or

$$\dot{p}(t) = \phi\left(\Delta_{\tau}\right)^{\alpha}\left[k(t-\tau)\right]^{\alpha} - (m+n)p(t). \tag{7}$$

Equations (6) and (7) construct the delay dynamic system of k and p,

$$\begin{aligned} \dot{k}(t) &= s\left(\Delta_{\tau}\right)^{\alpha}\left[k(t-\tau)\right]^{\alpha} - \delta\Delta_{\tau}k(t-\tau) - nk(t) - \theta p(t), \\ \dot{p}(t) &= \phi\left(\Delta_{\tau}\right)^{\alpha}\left[k(t-\tau)\right]^{\alpha} - (m+n)p(t). \end{aligned} \tag{8}$$

Notice that each equation has a delay τ and delay-dependent parameters.

We now determine the steady state of the system (8). Solving $\dot{p}(t) = 0$ for p yields the steady-state of p,

$$p^{*} = \frac{\phi\left(\Delta_{\tau}\right)^{\alpha}}{m+n}\left(k^{*}\right)^{\alpha}. \tag{9}$$

This is then substituted into the first equation of (8) to obtain the steady-state of k,

$$k^{*} = \left[\frac{\left(\Delta_{\tau}\right)^{\alpha}\left(s(m+n) - \theta\phi\right)}{\left(n + \delta\Delta_{\tau}\right)(m+n)}\right]^{\frac{1}{1-\alpha}}. \tag{10}$$

The sign of $s(m+n) - \theta\phi$ is ambiguous but it is positive if θ is sufficiently small. The signs of $n + \delta\Delta_{\tau}$ and $m + n$ are also ambiguous when $n < 0$. From an economic viewpoint, the depreciation rate and the pollution decay rate are probably larger than the growth rate in the absolute value (i.e., $\delta > |n|$ and $m > |n|$). Since $\Delta_{\tau} > 1$ for $n < 0$, $n + \delta\Delta_{\tau} > 0$ is a plausible consequence. To formally prove the steady-state capital positivity, we make the following assumption:

4） Guerrini (2012) and Varszegi (2018) employ the same transformation to a delay Solow model.

Assumption 1. (i) $s(m+n) - \theta\phi > 0$, (ii) $\min[m, \delta] > |n|$ for $n < 0$.

We now turn our attention to the stability of the steady state. By linearizing system (8) at the steady state and using the transformation, $x = k - k^*$ and $y = p - p^*$, we obtain a linear version of the dynamic system,

$$\begin{aligned} \dot{x}(t) &= (s\alpha Q - \delta\Delta_\tau)\, x(t-\tau) - nx(t) - \theta y(t), \\ \dot{y}(t) &= \alpha\phi Q x(t-\tau) - (m+n) y(t). \end{aligned} \tag{11}$$

with

$$Q = \frac{(n + \delta\Delta_\tau)\,(m+n)}{s(m+n) - \theta\phi}.$$

First, we consider two special cases in which (i) no delay (i.e., $\tau = 0$) and (ii) no pollution effect (i.e., $\theta = 0$). Then we proceed to the general system (11) with $\tau > 0$ and $\theta > 0$.

2.2 No Delay Model

Notice that $\Delta_\tau = 1$ for $\tau = 0$. The dynamic model (8) with $\tau = 0$ leads to a no-delay model consisting of two ordinary differential equations,

$$\begin{aligned} \dot{k}(t) &= sk^\alpha(t) - (n+\delta)\, k(t) - \theta p(t), \\ \dot{p}(t) &= \phi k^\alpha(t) - (m+n)\, p(t). \end{aligned} \tag{12}$$

The steady-states of (9) and (10) with $\Delta_\tau = 1$ are

$$p_n^* = \frac{\phi}{m+n}(k_n^*)^\alpha$$

and

$$k_n^* = \left(\frac{s(m+n) - \theta\phi}{(\delta+n)(m+n)} \right)^{\frac{1}{1-\alpha}}.$$

The dynamic system (11) with $\Delta_\tau = 1$ is the corresponding linear system of (12),

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \alpha s\bar{Q} - N & \theta \\ \alpha\phi\bar{Q} & -M \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{13}$$

where

$$\bar{Q} = \frac{MN}{sM - \theta\phi}, \quad M = m + n \text{ and } N = n + \delta.$$

For notational simplicity, we define $x = k - k_n^*$ and $y = p - p_n^*$. Let $\boldsymbol{J}$ be the coefficient matrix of system (13). The stability conditions are $tr\boldsymbol{J} < 0$ and $det\boldsymbol{J} > 0$, where

$$tr\boldsymbol{J} = \alpha s\bar{Q} - (M+N),$$

and

$$det\boldsymbol{J} = (1-\alpha)MN.$$

The determinant is always positive. However, the sign of $tr\boldsymbol{J}$ is ambiguous. We can rewrite it as

$$tr\boldsymbol{J} = -\frac{M+N}{Ms-\theta\phi}\left[\left(1-\alpha\frac{N}{M+N}\right)sM-\theta\phi\right]. \tag{14}$$

To guarantee the stability of the steady state in the non-delay model (12), we impose the following condition which holds when the value of θ is sufficiently small:

Assumption 2. $(1-\alpha)sM > \theta\phi$

Since Assumption 2 is stronger than Assumption 1(i) for $0 < \alpha < 1$, we can eliminate Assumption 1(i) when imposing Assumption 2,

$$sM > (1-\alpha)sM > \theta\phi \Longrightarrow sM - \theta\phi > 0.$$

This is Assumption 1(i) which assures $\bar{Q} > 0$ with Assumption 1(ii). We also have

$$\left(1-\alpha\frac{N}{M+N}\right) > (1-\alpha) \Longrightarrow \left(1-\alpha\frac{N}{M+N}\right)sM > \theta\phi.$$

Due to this last inequality, the square-bracketed factor of (14) is positive. The first factor, $(\mathrm{M}+\mathrm{N})/(Ms-\theta\phi)$ is also positive. Therefore, the right-hand side of (14) is negative under Assumption 2. Therefore, we have $tr\boldsymbol{J} < 0$ and $det\boldsymbol{J} > 0$, indicating the stability of the steady-state:

Theorem 1 *Given Assumptions 1(ii) and 2, the steady state* (k_n^*, p_n^*) *of the non-delay dynamic system (12) is locally asymptotically stable.*

We have assumed no technical progress (i.e., $\dot{A}(t) = 0$). If we further assume that pollution does not affect capital accumulation (i.e., $\theta = 0$), then system (12) reduces to the simplified system considered by Xepapades (2005),

$$\begin{aligned} \dot{k}(t) &= sk^{\alpha}(t) - (n+\delta)k(t), \\ \dot{p}(t) &= \phi k^{\alpha}(t) - mp(t). \end{aligned} \tag{15}$$

Assuming $\theta = 0$ yields two results: first, Assumption 2 always holds, and second, $\bar{Q} = (n+\delta)/s$. If we denote the coefficient matrix of system (15) by $\boldsymbol{J}_{\theta}$, then

$$tr\boldsymbol{J}_{\theta} = -(1-\alpha)(n+\delta) - m < 0 \text{ and } det\boldsymbol{J}_{\theta} = (1-\alpha)(n+\delta)m > 0.$$

These conditions imply that a unique nontrivial steady-state of system (15) is locally asymptotically stable for nonnegative initial conditions, although Theorem 1 has already established the stability of system (15).

3 Delay Solow Model with $\theta = 0$

When $\theta = 0$, the system (8) reduces to

$$\begin{aligned} \dot{k}(t) &= s\left(\Delta_\tau\right)^\alpha \left[k(t-\tau)\right]^\alpha - \delta\Delta_\tau k(t-\tau) - nk(t), \\ \dot{p}(t) &= \phi\left(\Delta_\tau\right)^\alpha \left[k(t-\tau)\right]^\alpha - (m+n)p(t). \end{aligned} \tag{16}$$

The steady-state values of k and p are (10) with $\theta = 0$ and (9), respecively:

$$k_\theta^* = \left(\frac{s\left(\Delta_\tau\right)^\alpha}{n + \delta\Delta_\tau}\right)^{\frac{1}{1-\alpha}}$$

and

$$p_\theta^* = \frac{\phi\left(\Delta_\tau\right)^\alpha}{m+n}\left(k^*\right)^\alpha.$$

The linearized version of system (16), evaluated at the steady state, is

$$\begin{aligned} \dot{x}(t) &= -\left[(1-\alpha)\,\delta\Delta_\tau - \alpha n\right]x(t-\tau) - nx(t), \\ \dot{y}(t) &= \alpha\phi\frac{n + \delta\Delta_\tau}{s}x(t-\tau) - (m+n)y(t) \end{aligned} \tag{17}$$

where, again, for notational simplicity, $x(t) = k(t) - k_\theta^*$ and $y(t) = p(t) - p_\theta^*$. For exponential solutions $x(t) = e^{\lambda t}u$ and $y(t) = e^{\lambda t}v$, the corresponding characteristic equation is

$$\det\begin{pmatrix} \lambda + n + \left[(1-\alpha)\,\delta\Delta_\tau - \alpha n\right]e^{-\lambda\tau} & 0 \\ -\alpha\phi\dfrac{n+\delta\Delta_\tau}{s}e^{-\lambda\tau} & \lambda + (m+n) \end{pmatrix} = 0$$

or equivalently,

$$\left\{\lambda + n + \left[(1-\alpha)\,\delta\Delta_\tau - \alpha n\right]e^{-\lambda\tau}\right\}\left\{\lambda + (m+n)\right\} = 0.$$

One solution is negative,

$$\lambda = -(m+n) < 0.$$

The other solutions solve

$$\lambda + n + c(\tau)e^{-\lambda\tau} = 0 \tag{18}$$

where

$$c(\tau) = (1-\alpha)\,\delta\Delta_\tau - \alpha n.$$

Since $\lambda = 0$ does not solve equation (18), we assume a purely imaginary root, $\lambda = i\omega$ with $\omega > 0$. Substituting $\lambda = i\omega$ into equation (18) and separating the real and imaginary parts yields

$$-n = c(\tau)\cos\omega\tau \text{ and } \omega = c(\tau)\sin\omega\tau. \tag{19}$$

Adding the squares of these two equations yields

$$\omega^2 = (c(\tau)+n)\,(c(\tau)-n)\,. \tag{20}$$

When $n > 0$, the first factor on the right-hand side of (20) is positive,

$$c(\tau)+n = (1-\alpha)\,(n+\delta\Delta_\tau) > 0,$$

and the sign of the second factor is ambiguous,

$$c(\tau)-n = (1-\alpha)\,\delta\Delta_\tau - (1+\alpha)\,n \gtreqless 0.$$

If $(1-\alpha)\delta\Delta_\tau - (1+\alpha)\,n \leq 0$, then $\omega^2 \leq 0$, and there is no $\lambda = i\omega$ with $\omega > 0$ satisfying (18). Therefore, there is no stability switching, and thus, the steady-state is locally asymptotically stable for any $\tau \geq 0$, meaning the delay is *harmless* to the stability of the model. Continuing the analysis, we assume the opposite inequality $c(\tau) - n > 0$, which leads to $\omega^2 > 0$:

Assumption 3. $(1-\alpha)\,\delta\Delta_\tau - (1+\alpha)\,n > 0$ when $n > 0$.

When $n < 0$, the first factor on the right-hand side of (20) is positive by Assumption 1(ii),

$$c(\tau)+n = (1-\alpha)\,(n+\delta\Delta_\tau) > 0.$$

The second factor is always positive,

$$c(\tau)-n = (1-\alpha)\,\delta\Delta_\tau + (1+\alpha)\,(-n) > 0.$$

Therefore, we again have $\omega^2 > 0$. From (19),

$$\sin\omega^*(\tau)\tau = \frac{\omega^*(\tau)}{c(\tau)} > 0 \text{ and } \cos\omega^*(\tau)\tau = -\frac{n}{c(\tau)}\begin{cases} < 0 \text{ if } n > 0 \\ > 0 \text{ if } n < 0 \end{cases}$$

where

$$\omega^*(\tau) = \sqrt{(c(\tau)+n)\,(c(\tau)-n)} \text{ and } 0 < \omega^*(\tau) < c(\tau),$$

$$c(\tau) > n > 0 \text{ for } n > 0 \text{ by Assumption 3}$$

and

$$c(\tau) > -n > 0 \text{ for } n < 0 \text{ by Assumption 1(ii).}$$

Regardless of whether $\cos\omega^*(\tau)\tau$ is positive or negative, $\sin\omega^*(\tau)\tau > 0$ implies that the threshold value τ^*_ℓ satisfies the following relation,

$$\tau^*_\ell = \frac{1}{\omega^*(\tau^*_\ell)}\left[\cos^{-1}\left(-\frac{n}{c(\tau^*_\ell)}\right) + 2\ell\pi\right] \text{ for } \ell = 0, 1, 2, \ldots \tag{21}$$

Equation (21) is an implicit function that can be solved numerically. Figures 1(A) and 1(B) show the same division of the (n, τ) region. However, for the sake of clarity in the following explanations, the ranges of the vertical and horizontal axes differ. Figure 1(A) with $0.004 \le n \le 0.008$ and $0 \le \tau \le 500$, shows the downward-sloping blue curve,

$$\tau = f(\tau) \equiv \frac{1}{n} \ln \left[\frac{(1-\alpha)\,\delta}{(1+\alpha)\,n} \right].$$

This is an alternative form of Assumption 3. In the light blue region above the blue curve, Assumption 3 does not hold. Therefore, the delay is harmless, and the steady state is locally asymptotically stable. The solid red and black dotted curves in the yellow region are the loci of (n, τ) that solve the second and first equations with $\omega^*(\tau)$ in (19), respectively. Note that the distorted inverse C-shaped part of the solid red curve and the dotted black curves overlap perfectly. This means that any point on this part is a solution of equation (21). Furthermore, the inverse C-shaped part indicates that equation (21) can have two roots for certain values of n, which we denote by $\tau_0^*(n)$ and $\tau_0^{**}(n)$. The vertical line at $\bar{n} \simeq 0.00727$,⁵⁾ is tangent to the inverse C-shaped curve. Thus equation (21) has equal roots,

$$\bar{\tau}_0 = \tau_0^*(n) = \tau_0^{**}(n) \simeq 90.755.$$

Therefore, it is possible to obtain two roots for $n < \bar{n}$.⁶⁾ For example, the vertical line at $n_0 = 0.5\%$ intersects the inverse C-shaped part twice, at which two distinct roots exist: The first and second roots at points a and a' are given by

$$\tau_a^* = \tau_0^*(n_0) \simeq 46.32 \text{ and } \tau_a^{**} = \tau_0^{**}(n) \simeq 296.24.$$

However, the larger root is excessively large and may lose its economic meaning. Accordingly, we focus our attention primarily on the smaller root. Figure 1(B) shows the region of interest for $-0.01 \le n \le 0.025$ and $0 \le \tau \le 50$. We divide this region into three subregions using two curves: The blue curve is $\tau = f(\tau)$ as in Figure 1(A), and the red curve is equation (21) with $\ell = 0$, which determines the minimum threshold value of τ_0^* against the value of n. As noted, the steady-state is locally stable in the light blue region. Below or to the left toof the blue curve, Assumption 3 holds. The red curve divides this region into two subregions. As will be shown later, a stability switche occurs along the red curve: the steady-state is stable in the yellow region and unstable in the light red region.

To determine the direction of the stability switch, we examine the transversality condition of equation (18). We will focus on the smaller value, $\tau_0^*(n)$, at which the stability might be lost for the first time. For the numerical analysis, we adopt the following parameter specification: $s = 0.2$, $\alpha = 0.5$, $\delta = 0.1$. We fFirst we determine the threshold values of τ at points, a, b, and c, for $n = n_0$, $n = 0$ and $n = -n_0$ where $n_0 = 0.005$ (0.5%),

$$\tau_a^* = \tau_0^*(n_0) \simeq 46.32, \ \tau_b^* = \tau_0^*(0) \simeq 31.42 \text{ and } \tau_c^* = \tau_0^*(-n_0) \simeq 25.19.$$

The term $\varepsilon_1(\tau)$, which is defined in equation (A-4) of the Appendix, is given by

$$\varepsilon_1(\tau) = K \left[k_0 + k_1 \delta \Delta_\tau + k_2 (\delta \Delta_\tau)^2 + k_3 (\delta \Delta_\tau)^3 \right]$$

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5) This approximated value is obtained by trial and error.

6) For a very small values, $n_1 = 0.0001$, we have that $\tau_0^*(n_1) \simeq 31.5874$ and $\tau_0^{**}(n_1) \simeq 57413.7$. For $n = 0$, there is only one toot, $\tau_b^* = \tau_0^*(0) \simeq 31.4157$. The loci of $\tau = \tau_0^*(n)$ may be asymptotic to the vertical axis.

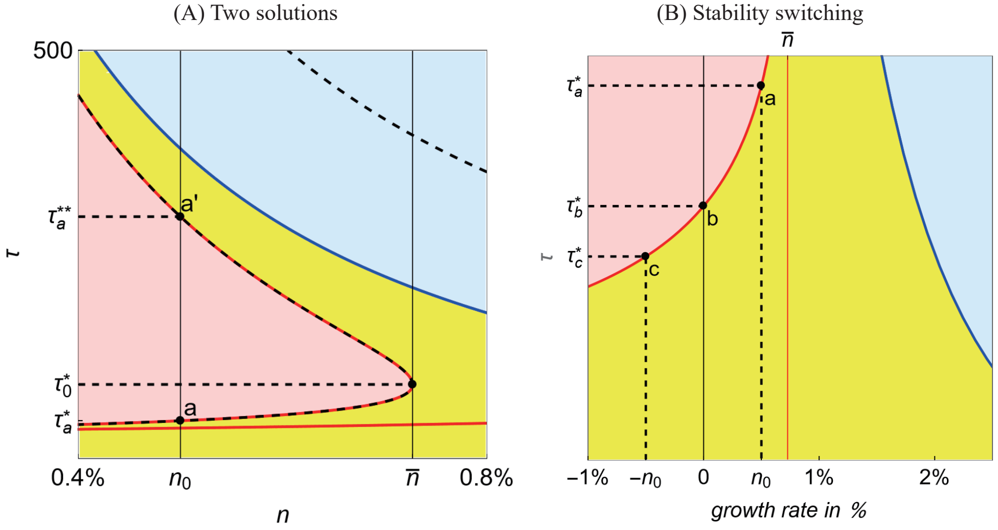


Figure 1. Division of the (n,τ) region.

(A) Two solutions

(B) Stability switching

with

$$K = (1-\alpha)\left[(1-\alpha)\delta\Delta_\tau - \alpha n\right],\ k_0 = \alpha(1+\alpha)n^3,$$

$$k_1 = -\left[2-(3-n\tau)\alpha^2\right]n^2,\ k_2 = -(1-\alpha)(3-2n\tau)\alpha n,\ k_3 = (1-\alpha)^2(1-n\tau).$$

Setting $n = n_0 (= 0.5\%)$ and solving $\varepsilon(\tau) = 0$ yields $\tau_\varepsilon \simeq 155.64$. Furthermore, it is numerically confirmed that $\varepsilon_1(\tau)$ is monotonically decreasing in τ. Hence, we have

$$\varepsilon(\tau) > 0 \text{ for } \tau < \tau_\varepsilon \text{ and } \varepsilon(\tau) < 0 \text{ for } \tau > \tau_\varepsilon,$$

implying

$$\varepsilon(\tau_a^*) > 0 \text{ as } \tau_a^* < \tau_\varepsilon.$$

It is also numerically verified that

$$\varepsilon_1(\tau) > 0 \text{ for any } \tau \geq 0 \text{ when } n = 0 \text{ and } n = -n_0.$$

Hence,

$$\varepsilon(\tau_b^*) > 0 \text{ and } \varepsilon(\tau_c^*) > 0.$$

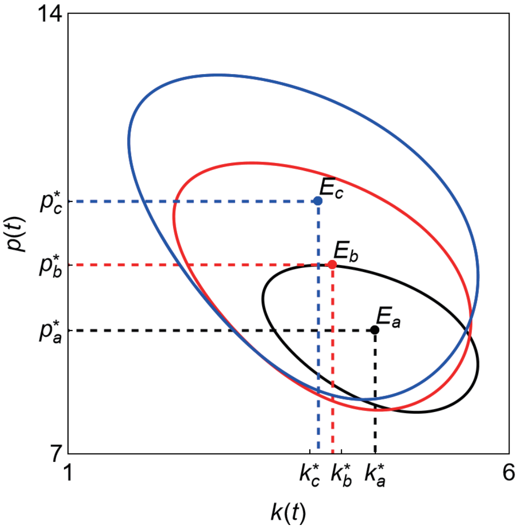
According to (A-5) in the Appendix, we have

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}_{|\lambda=i\omega(\tau)}\right] > 0 \text{ for } \tau = \tau_a^*,\ \tau_b^*,\ \tau_c^*.$$

The direction of this inequality indicates that the real part crosses the imaginary axis from left to right at each threshold value. Therefore, stability is lost in the light red region in which $\tau > \tau_i^*$ for $i = a, b, c$, if the value of τ increases along the vertical line at $n = n_0$, 0, or $-n_0$. Figure 2 shows the formation of three limit cycles surrounding the corresponding steady states when the value of τ exceeds the corresponding threshold value,

$$\tau = \tau_a^* + 1 \text{ for } n = n_0,\ \tau = \tau_b^* + 1 \text{ for } n = 0 \text{ and } \tau = \tau_c^* + 1 \text{ for } n = -n_0.$$

Figure 2. Birth of limit cycles



In summary, we have the following result:

Theorem 2 *Suppose that Assumption 3 holds. Then the steady-state,* (k_θ^*, p_θ^*), *is locally asymptotically stable for* $\tau > \tau_0^*(n)$ *and loses stability for* $\tau \geq \tau_0^*(n)$ *when* $n < \bar{n}$. *In contrast, it is always locally asymptotically stable regardless of the value of* τ *when* $n > \bar{n}$.

When $n = 0$, our model reduces to that of Ferrara et al. (2014),

$$\begin{aligned} \dot{k}(t) &= s\,[k(t-\tau)]^\alpha - \delta k(t-\tau), \\ \dot{p}(t) &= \phi\,[k(t-\tau)]^\alpha - mp(t). \end{aligned} \tag{22}$$

First, it has already been shown that a Hopf bifurcation occurs as the delay passes through the threshold value,[7]

$$\bar{\tau} = \frac{\pi}{2(1-\alpha)\delta} \simeq 31.42,$$

Second, it has been shown that the direction analysis, normal form theory and the center manifold theorem verify the stability criteria of the bifurcating periodic solution. The characteristic equation (18) with $n = 0$ reduces to

$$\lambda + (1-\alpha)\delta e^{-\lambda\tau} = 0.$$

We can obtain the threshold value $\bar{\tau}$ by applying Theorem 1 of Hayes (1950) to this characteristic equation. Additionally, we can also numerically verify the occurrence of the stability switch at $\bar{\tau} = \tau_b^*$ through a supercritical Hopf bifurcation.

7） Notice that their $\bar{\tau}$ is identical with our τ_b^*.

4 Delay Solow Model with $\theta > 0$

In this section, we assume that $\theta > 0$ and address the stability of the positive steady state (k^*, p^*) of the system (8). To this end, we examine the dynamic structure around the zero steady state of system (11). The characteristic equation with exponential solutions, $x(t) = e^{\lambda t}u$ and $y(t) = e^{\lambda t}v$, is

$$\det\begin{pmatrix} \lambda + n - (s\alpha Q - \delta\Delta_\tau)\, e^{-\lambda\tau} & \theta \\ \alpha\phi Q e^{-\lambda\tau} & \lambda + M \end{pmatrix} = 0.$$

The expanded form is quadratic in λ,

$$\lambda^2 + \left[b_1 + b_2(\tau)e^{-\lambda\tau}\right]\lambda + \left[c_1 + c_2(\tau)e^{-\lambda\tau}\right] = 0 \tag{23}$$

where

$$b_1 = M + n,\ b_2(\tau) = \frac{\left[(1-\alpha)\delta\Delta_\tau - \alpha n\right] Ms - \delta\Delta_\tau\theta\phi}{Ms - \theta\phi}$$

and

$$c_1 = Mn,\ c_2(\tau) = M\left[(1-\alpha)\delta\Delta_\tau - \alpha n\right].$$

Equation (23) with $\lambda = 0$ is reduced to

$$c_1 + c_2(\tau) = M(1-\alpha)\,(n + \delta\Delta_\tau) > 0. \tag{24}$$

This inequality implies that $\lambda = 0$ is not a solution of equation (23). Assuming next $\lambda = i\omega$ with $\omega > 0$ and substituting this solution into (23), we can divide the resultant characteristic equation into the real and imaginary parts,

$$\begin{aligned} \omega b_2(\tau)\sin\omega\tau + c_2(\tau)\cos\omega\tau &= \omega^2 - c_1, \\ \omega b_2(\tau)\cos\omega\tau - c_2(\tau)\sin\omega\tau &= \omega b_1. \end{aligned} \tag{25}$$

Adding the squares of these equations yields a biquadratic equation in ω (i.e., a quadratic equation in ω^2)

$$\omega^4 - \left(2c_1 + b_2^2(\tau) - b_1^2\right)\omega^2 + c_1^2 - c_2^2(\tau) = 0. \tag{26}$$

Since Assumption 3 implies

$$c_1 - c_2(\tau) = -M\left[(1-\alpha)\,\delta\Delta_\tau - (1+\alpha)n\right] < 0,$$

This last inequality and equation (24) imply that the constant term of equation (23) is negative. Consequently, equation (26) admits a positive root,

$$\omega_*^2(\tau) = \frac{2c_1 + b_2^2(\tau) - b_1^2 + \sqrt{D}}{2} > 0, \tag{27}$$

where the discriminant is positive,

$$D = \left(2c_1 + b_2^2(\tau) - b_1^2\right)^2 - 4\left(c_1^2 - c_2^2(\tau)\right) > 0.$$

Replacing ω with ω_* and solving the two equations in (25) yield two solutions the following expressions:

$$\sin\omega_*(\tau)\tau = \frac{\omega_*(\tau)b_2(\tau)\left(\omega_*^2(\tau) - c_1\right) + b_1 c_2(\tau)}{\omega_*^2(\tau)b_2^2(\tau) + c_2^2(\tau)}, \tag{28}$$

and

$$\cos\omega_*(\tau)\tau = \frac{\left[c_2(\tau) - b_1 b_2(\tau)\right]\omega_*^2(\tau) - c_1 c_2(\tau)}{\omega_*^2(\tau)b_2^2(\tau) + c_2^2(\tau)}. \tag{29}$$

To determine the threshold value of τ, we must first identify the signs of $\sin\omega_*(\tau)\tau$ and $\cos\omega_*(\tau)\tau$. However, these trigonometric functions are too complex to be evaluated analytically. For this reason, we specify the parameters and examine their values numerically. To proceed, we carefully address Assumptions 2 and 3: Assumption 2 ensures the positivity of the steady-state per capita capital and the stability of the non-delay model. Assumption 3 is the key condition verifying the occurence of a stability switch in the delay Solow model with $\theta = 0$. We specify the parameter values as follows

$$\alpha = 0.5,\ \delta = 0.1,\ \theta = 0.025,\ \phi = 0.3,\ s = 0.2,\ m{=}0.1. \tag{30}$$

Numerical calculations confirm that the numerator of equation (28) is positive for $-0.001 \leq n \leq 0.02$ and $0 \leq \tau \leq 50$. Therefore, the threshold value τ_* satisfies the following,

$$\tau_* = \frac{1}{\omega_*(\tau_*)}\cos^{-1}\left(\frac{\left[c_2(\tau_*) - b_1 b_2(\tau_*)\right]\omega_*^2(\tau_*) - c_1 c_2(\tau_*)}{\omega_*^2(\tau_*)b_2^2(\tau_*) + (c_2^2(\tau_*)}\right). \tag{31}$$

Figure 2(A) is similar to Figure 1(A) in that it divides the (n, τ) region into the stability and instability region. The stability region is a union of the yellow and the light blue regions while the instability region is the light red region. As will be seen shortly, the stability switches to instability when the delay increases along the vertical dotted line.

The red curve has an inverse C-shaped profile; however, its positively sloped segment is displayed in Figure 2(A). For $\bar{n} \simeq 0.00752$, equation (31) generates equal roots,

$$\bar{\tau}_0 = \tau_0^*(\bar{n}) = \tau_0^{**}(\bar{n}) \simeq 84.914.$$

For $n < \bar{n}$, we focus only on the smaller root, even though equation (31) has two roots. We check the transversality condition of the characteristic equation (23). As before, we determine the threshold values of τ at points a, b and c for $n = n_0$, $n = 0$ and $n = -n_0$ where $n_0 = 0.005$,

$$\tau_a^* = \tau^*(n_0) \simeq 41.889,\ \tau_b^* = \tau^*(0) \simeq 28.613,\ \tau_c^* = \tau^*(-n_0) \simeq 22.775.$$

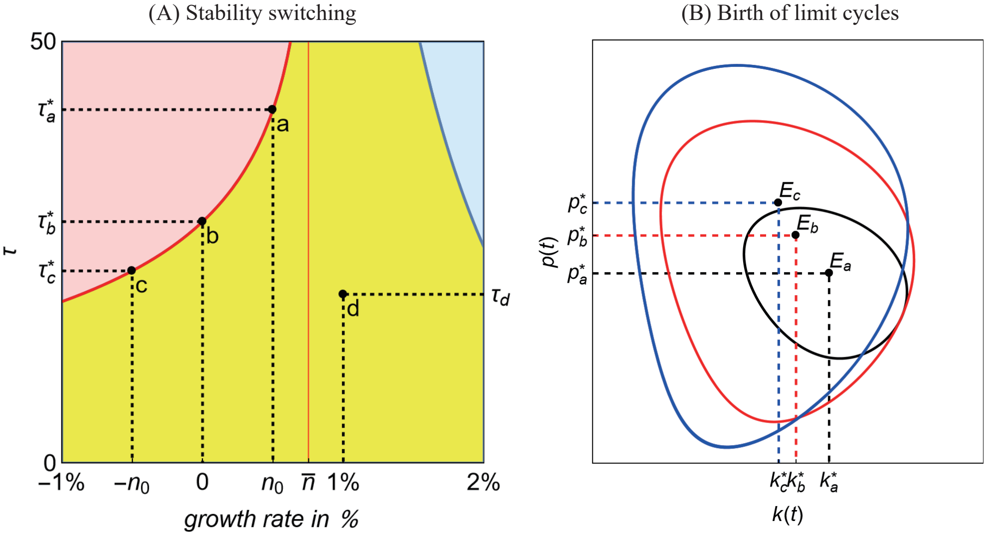
The quantity $\varepsilon_2(\tau)$, defined in (B-6) in the Appendix, is calculated as

$$\varepsilon_2^a = \varepsilon_2(\tau_a^*) \simeq 1.6504 \times 10^{-8} > 0,$$

$$\varepsilon_2^b = \varepsilon_2(\tau_b^*) \simeq 5.5094 \times 10^{-8} > 0,$$

$$\varepsilon_2^c = \varepsilon_2(\tau_c^*) \simeq 9.4467 \times 10^{-8} > 0.$$

Figure 3. Delay dynamics with $\theta > 0$


(A) Stability switching

(B) Birth of limit cycles

Hence, from (B-7) in the Appendix, we obtain

$$\left(\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega(\tau_i^*)}\right]\right)=\varepsilon_2^i>0 \text{ for } i=a,b,c.$$

This inequality implies that the real part crosses the imaginary axis from left to right at each threshold value. Therefore, stability is lost in the light red region in which $\tau > \tau_i^*$ for $i = a, b, c$. Figure 2(B) shows the birth of limit cycles for the following values of τ

$$\tau=\tau_a^*+1 \text{ for } n=n_0,\ \ \tau=\tau_b^*+1 \text{ for } n=0, \text{ and } \tau=\tau_c^*+0.4 \text{ for } n=-n_0.$$

We now observe the same dynamics from a different perspective. Figure 4(A)[8] depicts the time trajectories of $\log[K(t)]$ with the same growth rate $n = n_0 = 0.5\%$ and the delay value, $\tau = 42 > \tau_a^*$, but with the different initial points,

the red curve is for $k_0 = k^* + 1$ and $p_0 = p^* + 1$,

the black curve is for $k_0 = k^*$ and $p_0 = p^*$,

the blue curve is for $k_0 = k^* - 1$ and $p_0 = p^* - 1$,

the dotted black represents of K^*e^{nt} with $K^* = k^*L_0$.

The upward-sloping dotted line in Figure 4(A) shows $logK^*e^{nt}$ versus t, and represents the balanced growth path. The dotted horizontal line in Figure 4(B) is the $\dot{K}(t)/K(t) = n_0$ line.

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[8] For the graphical convinecе, we define $\hat{K} = \log K$, $\hat{K}_0^A = \log K_0^A$ and $\hat{K}_0^B = \log K_0^B$.

Figure 4. Time trajectories of the capital stock and its growth rate for $n < \bar{n}$

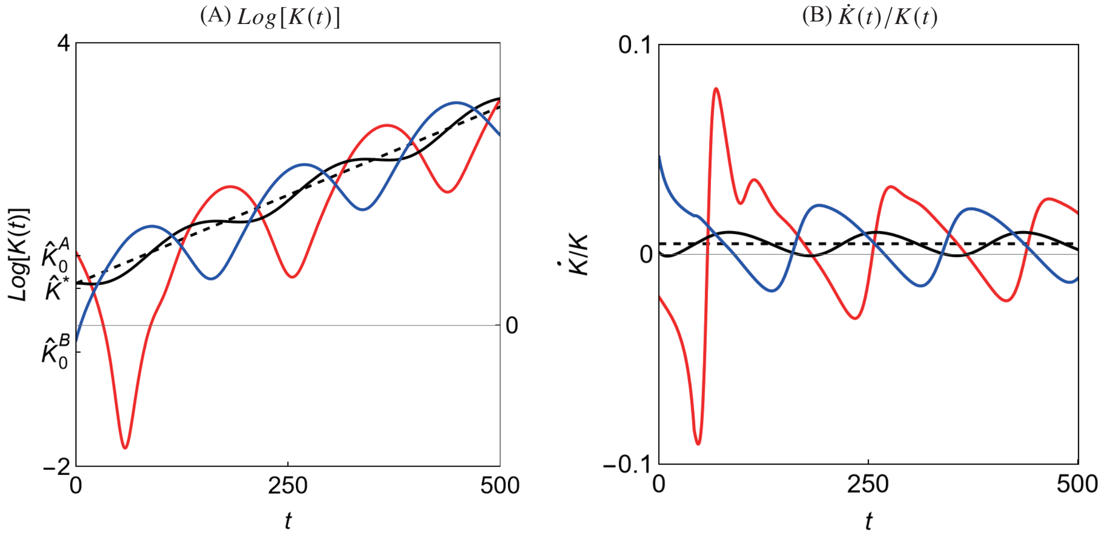
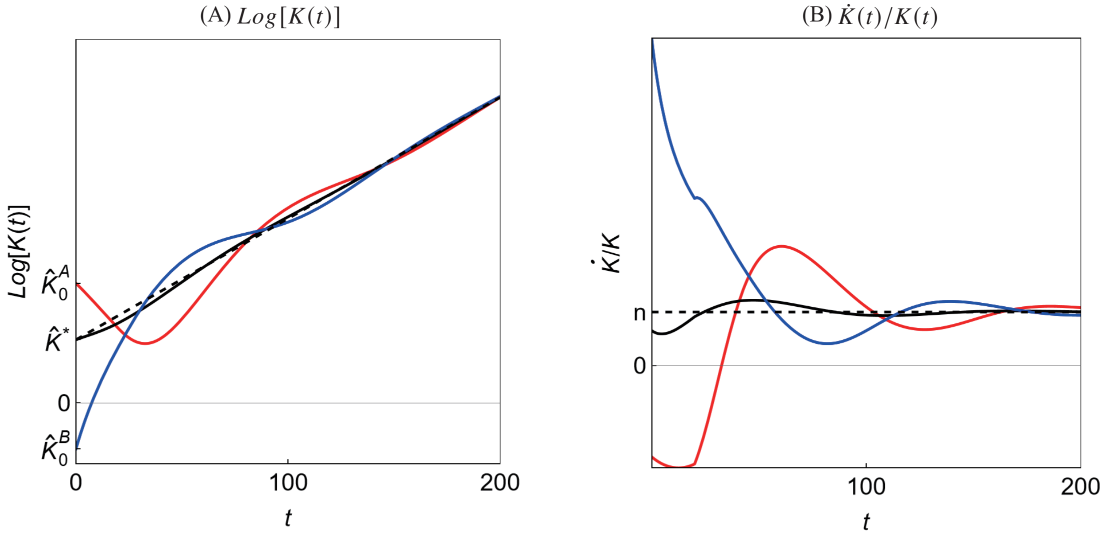

(A) $Log[K(t)]$ (B) $\dot{K}(t)/K(t)$

Figure 5. Time trajectories of the capital stock and its growth rate for $n > \bar{n}$


(A) $Log[K(t)]$ (B) $\dot{K}(t)/K(t)$

In Figure 5(A), we take point d with $n_d = 0.01 > \bar{n}$ and $\tau_d = 20$. This point lies in the yellow region of Figure 3(A), and the corresponding steady states of $k^*(n_d, \tau_d)$ and $p^*(n_d, \tau_d)$ are locally asymptotically stable, and the characteristic equation has a pair of complex roots. Therefore, the time trajectories of $K(t)$ oscillate and converge to a balanced growth path, which is represented by the black dotted curve. The growth rate also oscillatory converges to the $\dot{K}(t)/K(t) = n_d$ line.

5 Concluding Remarks

This paper investigates whether the long-run equilibrium remains stable if the time-to-build production delays affect the capital and pollution accumulation under non-zero population growth. There are two answers to this question, both of which depend critically on the population growth rate and the length of delay. Both the population growth rate and the delay length have threshold values. Our findings are as follows:

(A1) If the population growth rate exceeds its threshold value, then the steady state is locally asymptotically stable regardless of the delay length.

(A2) If the population growth rate is less than the threshold, the stedy-state loses stability and gives rise to a limit cycle through a Hopf bifurcation if the delay exceeds the threshold.

We also consider the impact of pollution on capital accumulation and arrive at the following conclusions. Regardless of whether pollution is present, the two results described above, (A1) and (A2), hold. However, the specific threshold values differ; the threshold value with the pollution effect is larger than the value without the pollution effect. This result implies that pollution narrows the area in which (A1) occurs. Long-run economic growth depends on the efforts to clean up the environment. If environmental care is favorable to the economy, future studies should extend our analysis to examine how to control existing pollution and abatement technology in order to optimize output.

Appendix

Following Beretta and Kuang (2003), this Appendix examines the transversality crossing conditions for the first-order and second-order characteristic equations ((18) and (23), respectively).

We reiterate equation (18),

$$\lambda + n + c(\tau)e^{-\lambda\tau} = 0 \tag{A-1}$$

We denote the derivative of $c(\tau)$ with respect to τ by $c'(\tau)$. Differentiating (A-1) with respect to τ yields

$$\left(1 - \tau c(\tau)e^{-\lambda\tau}\right)\frac{d\lambda}{d\tau} - \left(\lambda c(\tau) - c'(\tau)\right)e^{-\lambda\tau} = 0.$$

It is convenient to consider $(d\lambda/d\tau_2)^{-1}$. Hence, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{e^{\lambda\tau} - \tau c(\tau)}{\lambda c(\tau) - c'(\tau)}. \tag{A-2}$$

The characteristic equation (A-1) can be rewritten as

$$e^{\lambda\tau} = -\frac{c(\tau)}{n + \lambda}. \tag{A-3}$$

Substituting (A-3) into (A-2) and evaluating the resultant expression at $\lambda = i\omega(\tau)$ with $\omega(\tau) > 0$, yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{|\lambda=i\omega(\tau)} = \frac{-\frac{c(\tau)[n-i\omega(\tau)]}{n^2+\omega^2(\tau)} - \tau c(\tau)}{ic(\tau)\omega(\tau) - c'(\tau)}.$$

The following holds for $\tau > 0$: $\omega^2(\tau) + n^2 = c^2(\tau)$, and its derivative relation, $\omega(\tau)\omega'(\tau) + b(\tau)b'(\tau) = c(\tau)c'(\tau)$.

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{|\lambda=i\omega(\tau)} = \frac{-\left[n + \tau c^2(\tau)\right] + i\omega(\tau)}{-\omega(\tau)\omega'(\tau) + i\omega(\tau)c^2(\tau)}.$$

Rationalizing the denominator and taking out only the real part, we have

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}_{|\lambda=i\omega(\tau)}\right] = \frac{\varepsilon_1(\tau)}{[\omega(\tau)\omega'(\tau)]^2 + \left[\omega(\tau)c^2(\tau)\right]^2}$$

where the denominator is positive, and the numerator is of the form,

$$\varepsilon_1(\tau) = \left[n + \tau c^2(\tau)\right]\omega(\tau)\omega'(\tau) + [\omega(\tau)c(\tau)]^2. \tag{A-4}$$

Hence,

$$sign\left\{\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}_{|\lambda=i\omega(\tau)}\right]\right\} = sign[\varepsilon_1(\tau)]. \tag{A-5}$$

This leads to the following:

Result A. *Purely imaginary roots of the characteristic equation (A-1), $\lambda(\tau) = i\omega(\tau)$ with $\omega(\tau) > 0$ and $\tau > 0$, cross the imaginary axis from left to right if $\varepsilon_1(\tau) > 0$ and from right to left if $\varepsilon_1(\tau) < 0$.*

We reiterate equation (23),

$$\lambda^2 + \left[b_1 + b_2(\tau)e^{-\lambda\tau}\right]\lambda + \left[c_1 + c_2(\tau)e^{-\lambda\tau}\right] = 0. \tag{B-1}$$

Differentiating equation (B-1) with respect to τ yields

$$\left[2\lambda + b_1 + \left(b_2(\tau) - b_2(\tau)\lambda\tau - c_2(\tau)\right)e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = -\left[b_2'(\tau)\lambda - b_2(\tau)\lambda^2 + c_2'(\tau) - c_2(\tau)\lambda\right]e^{-\lambda\tau}.$$

For convenience, we consider $(d\lambda/d\tau)^{-1}$,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + b_1 + \left[b_2(\tau) - \left(b_2(\tau)\lambda + c_2(\tau)\right)\tau\right]e^{-\lambda\tau}}{\left[b_2(\tau)\lambda + c_2(\tau)\right]\lambda - \left[b_2'(\tau)\lambda + c_2'(\tau)\right]e^{-\lambda\tau}}. \tag{B-2}$$

The characteristic equation is solved for $e^{-\lambda\tau}$,

$$e^{-\lambda\tau} = -\frac{\lambda^2 + b_1\lambda + c_1}{b_2(\tau)\lambda + c_2(\tau)}.$$

This is substituted into (B-2),

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-\dfrac{2\lambda + b_1}{\lambda^2 + b_1\lambda + c_1} + \dfrac{b_2(\tau)}{b_2(\tau)\lambda + c_2(\tau)} - \tau}{\lambda - \dfrac{b_2'(\tau)\lambda + c_2'(\tau)}{b_2(\tau)\lambda + c_2(\tau)}}.$$

We evaluate this expression at $\lambda = i\omega_*$ where $\omega_* > 0$ is obtained in (28) and the dependency on τ is temporarily omitted for notational simplicity,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega} = \frac{-\dfrac{b_1 + i2\omega}{(c_1 - \omega^2) + ib_1\omega} + \dfrac{b_2(\tau)}{c_2(\tau) + ib_2(\tau)\omega} - \tau}{i\omega - \dfrac{b_2'(\tau)\lambda + c_2'(\tau)}{c_2(\tau) + ib_2(\tau)\omega}}. \tag{B-3}$$

Rationalizing the denominator of each fraction and using the following relation obtained from equation (27)

$$\left(\omega^2 - c_1\right)^2 + (b_1\omega)^2 = c_2^2(\tau) + b_2^2(\tau)\omega^2,$$

we can arrange the form of (B-3) as

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega} = \frac{A_R + iA_I}{B_R + iB_I}$$

where

$$A_R = b_1\left(\omega^2 - c_1\right) - 2b_1\omega^2 + b_2(\tau)c_2(\tau) - \left[c_2^2(\tau) + b_2^2(\tau)\omega^2\right]\tau,$$

$$A_I = 2\omega\left(\omega^2 - c_1\right) + \left[b_1^2 - b_2^2(\tau)\right]\omega,$$

$$B_R = -\left[b_2(\tau)b_2'(\tau)\omega^2 + c_2(\tau)c_2'(\tau)\right],$$

$$B_I = \left[b_2(\tau)c_2'(\tau) - b_2'(\tau)c_2'(\tau)\right]\omega + \left[c_2^2(\tau) + b_2^2(\tau)\omega^2\right]\omega.$$

Taking out the real part, we have

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega}\right] = \frac{A_R B_R + A_I B_I}{B_R^2 + B_I^2}. \tag{B-4}$$

We use the following two relations (B-5) and (B-6) to organize the form of the numerator. Differentiating equation (27) with respect to τ yields

$$\omega(\tau)\omega'(\tau)\left[2\omega^2 - \left(2c_1 + b_2^2(\tau) - b_1^2\right)\right] = b_2(\tau)b_2'(\tau)\omega^2 + c_2(\tau)c_2'(\tau). \tag{B-5}$$

From (28), we have

$$2\omega_*^2 - \left(2c_1 + b_2^2(\tau) - b_1^2\right) = \sqrt{D} > 0.$$

Then $A_R B_R + A_I B_I$ is represented as

$$A_R B_R + A_I B_I = \sqrt{D}\varepsilon_2(\tau)$$

where

$$\varepsilon_2(\tau) = \left(\omega(\tau)^2 X(\tau) + \omega(\tau)\omega'(\tau)Y(\tau)\right) \tag{B-6}$$

$$X(\tau) = c_2^2(\tau) + b_2^2(\tau)\omega(\tau)^2 - b_2'(\tau)c_2(\tau) - b_2(\tau)c_2'(\tau)$$

and

$$Y(\tau) = -b_2(\tau)c_2(\tau) + \left[c_2^2(\tau) + b_2^2(\tau)\omega^2(\tau)\right]\tau + b_1(c_1 + \omega^2(\tau)).$$

Therefore, we finally arrive at the following:

$$sign\left(\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\lambda=i\omega}\right]\right) = sign\left[\varepsilon_2(\tau)\right]. \tag{B-7}$$

Result B. *The purely imaginary roots of the characteristic equation (B-1) are* $\lambda_\pm(\tau) = \pm i\omega(\tau)$ *with* $\omega(\tau) > 0$ *and* $\tau > 0$*, cross the imaginary axis from left to right when* $\varepsilon_2(\tau) > 0$ *and right to left when* $\varepsilon_2(\tau) < 0$.

Declaration

The authors declare that they have no conflicts of interest.

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