

Strange Attractors in an Open Economy Model with External Forcing

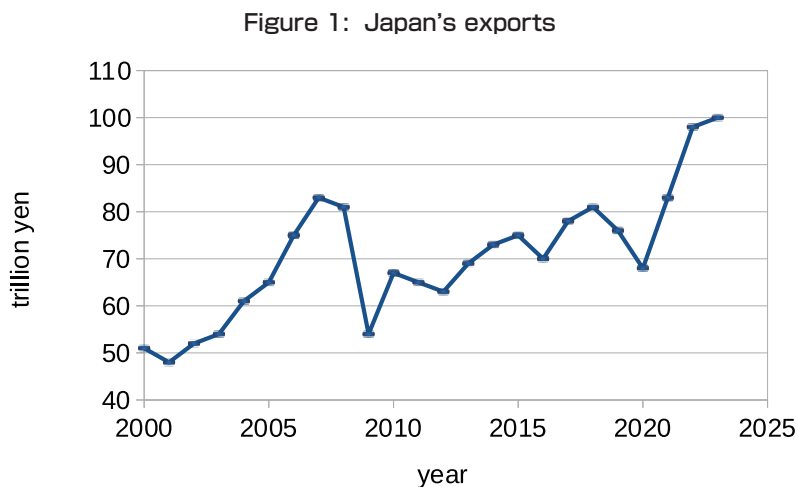
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1 Introduction

The global economy continues to gain momentum and evolve. As illustrated in Figure 1, Japan's exports have shown a generally upward trajectory, albeit with fluctuations, in parallel with the advancement of globalization. Looking more closely, the figures dropped significantly during the Lehman Shock and the COVID-19 crisis, but during other periods, they showed an upward trend. Although not shown in this figure, Japan's imports, like its exports, are also on a steady upward trend.

Within the standard framework of international macroeconomics, domestic exports are understood to be dependent on the GDP of foreign economies. This indicates that the performance of domestic economy is, to a considerable degree, dependent on the economic conditions prevailing in the foreign countries. Furthermore, recently, armed conflicts and climate change have been impacting economic activity in the world. These external shocks have contributed to increased instability in the global markets, thereby introducing greater volatility into global economic activity.

The purpose of the present paper is to examine the emergence of business cycles within the framework of a dynamic open-economy model. In particular, we seek to demonstrate that chaotic fluctuations in the economy can



arise through the application of chaos theory. We construct an open-economy model based on the Keynesian cross analysis and investigate its dynamic properties, incorporating a Kaldor-type investment function. This implies that we investigate our model by using the theory of nonlinear dynamical systems.

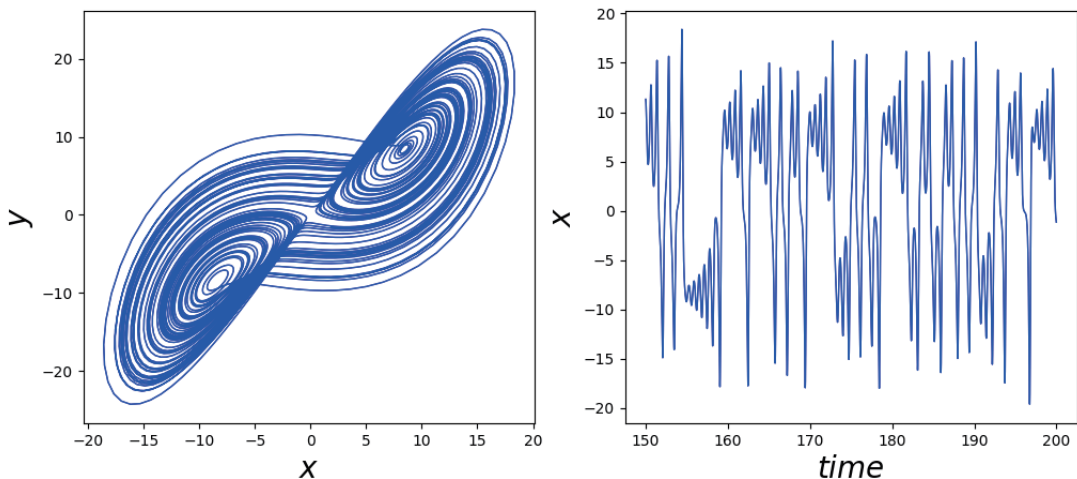
The present paper is structured as follows. Section 2 provides an overview of relevant discussions on chaos. Section 3 develops the Keynesian cross framework with consideration of a Kaldor-type investment function. Section 4 presents and analyzes a small open economy model. Section 5 discusses a comparison with the Lorenz (1993) model. Section 6 offers concluding remarks.

2 Three types of chaos in continuous time

This section focuses primarily on the analysis of chaos in continuous-time dynamical systems. While we do not explicitly address chaotic behavior in discrete-time systems, it is important to acknowledge that foundational contributions by Li and Yorke (1975), May (1976), and Sharkovsky (1964) significantly advanced the field of chaos theory. In particular, Li and Yorke's seminal work introduced the term “chaos” into the mathematical literature and demonstrated that even simple one-dimensional maps could exhibit highly complex and unpredictable behavior. May's biological models further illustrated how such dynamics could emerge in population systems, emphasizing the real-world relevance of chaotic phenomena. Sharkovsky's ordering theorem (1964) provided an important mathematical framework for understanding periodicity and its relationship to chaos in discrete maps. Although our focus remains on continuous-time formulations, these discrete-time contributions form a crucial part of the theoretical foundation upon which modern chaos theory is built.

We are now ready to examine three representative models that demonstrate chaos in continuous time. We begin with the Lorenz (1963) model. Lorenz, originally a meteorologist, investigated a system of ordinary differential equations in his seminal paper. He uncovered the phenomenon of sensitive dependence on initial conditions through his work on numerical weather simulations, a discovery that later became widely known as the butterfly effect in the winter of 1961. He demonstrated that even small differences in initial values could lead to dramatically

Figure 2: Lorenz chaos



divergent outcomes, thereby highlighting the intrinsic limitations of long-term weather forecasting in deterministic systems. The resulting Lorenz system consists of the following three equations:

$$\dot{x} = -\sigma x + \sigma y, \tag{1a}$$

$$\dot{y} = rx - y - xz, \tag{1b}$$

$$\dot{z} = xy - \beta z, \tag{1c}$$

where σ , r , and β are parameters. Despite its algebraic simplicity, only two nonlinear terms appear on the right-hand side (xz and xy) – the system exhibits surprisingly complex and irregular behavior, defying the expectations of the time. Figure 2 illustrates the Lorenz attractor, generated under the parameter values $\sigma = 10$, $r = 28$, and $\beta = 8/3$.

An important point to emphasize is that his discovery of sensitive dependence on initial conditions was achieved in the winter of 1961. On this point, see Gleick (1987). In this book, he presented Lorenz's 1961 printouts regarding computer simulations.

Next, we deal with the Rössler (1976) model. The Rössler system is also a well-known system that produces chaotic motions in continuous time. He investigated the following system:

$$\dot{x} = -y - z, \tag{2a}$$

$$\dot{y} = x + ay, \tag{2b}$$

$$\dot{z} = b + z(x - c), \tag{2c}$$

where a , b , and c are parameters. It is worth noting that, compared with the Lorenz system, only one quadratic term appears in the Rössler system, which contains the xz term on the right-hand side of Equation (2c). In this sense, the Rössler system possesses a structurally simpler form. Despite this relative simplicity, the system is capable of generating complex dynamical behavior. In particular, when the parameter values are set to $(a, b, c) = (0.45, 2, 4)$, the system produces the well-known Rössler attractor, a chaotic attractor whose structure is illustrated in Figure 3. This attractor is notable for its spiral-like geometry and serves as a canonical example of continuous-time chaos in low-dimensional systems.

Figure 3: Rössler chaos

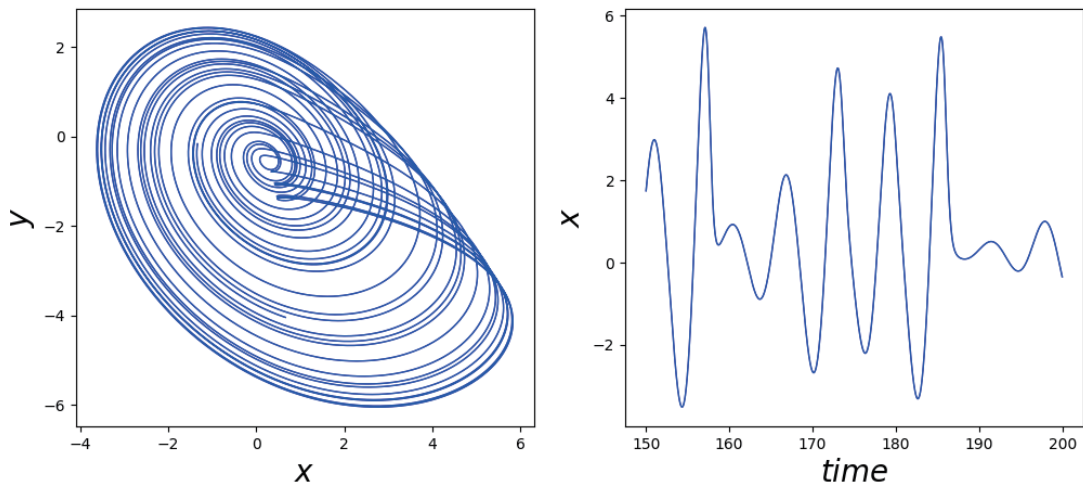
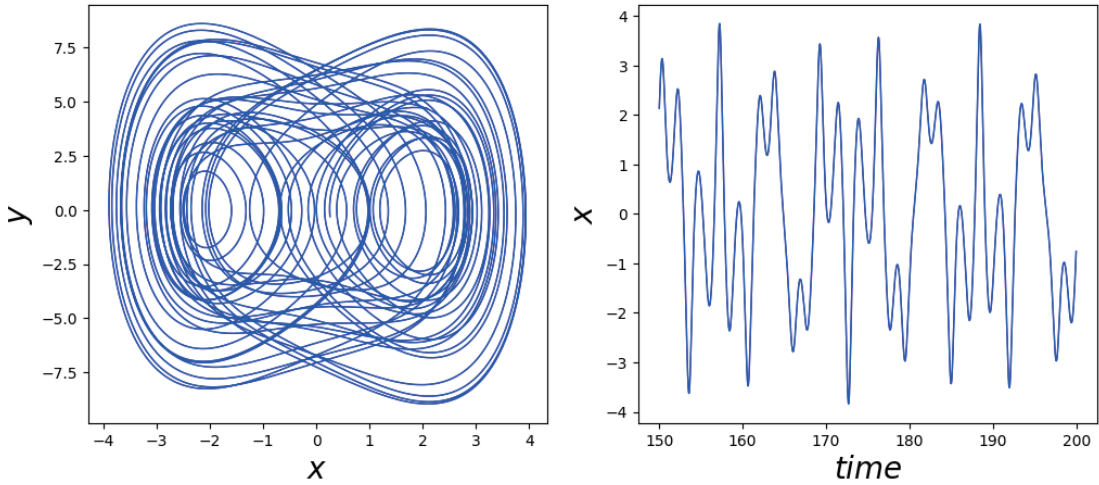


Figure 4: Ueda attractor



Lastly, we examine the Ueda model, which represents the third example of continuous-time chaotic dynamics. Yoshisuke Ueda is widely recognized as one of the pioneering researchers in the development of chaos theory. In his early work, he observed a strange attractor in experiments with electrical circuits. By investigating a non-autonomous system of differential equations with periodic excitation through the use of analog oscilloscopes, he identified a pattern indicative of chaotic behavior. This discovery, made on November 27, 1961, marked a seminal moment in the study of chaos theory, as detailed in Ueda (1992).

Ueda has investigated several types of the Duffing equation; here, we will specifically highlight one particular form as an illustrative example in Ueda (1991):

$$\dot{x} = y, \tag{3a}$$

$$\dot{y} = -ky - x^3 + B \cos t, \tag{3b}$$

where k and B are parameters. When the parameter values are chosen as $(k, B) = (0.1, 10)$, we obtain a strange attractor, illustrated in Figure 4. The left panel of Figure 4 depicts the strange attractor in the (x, y) -plane. The right panel presents the time path of $x(t)$ over the interval $150 \leq t \leq 200$.

3 Keynesian cross with Kaldorian investment function

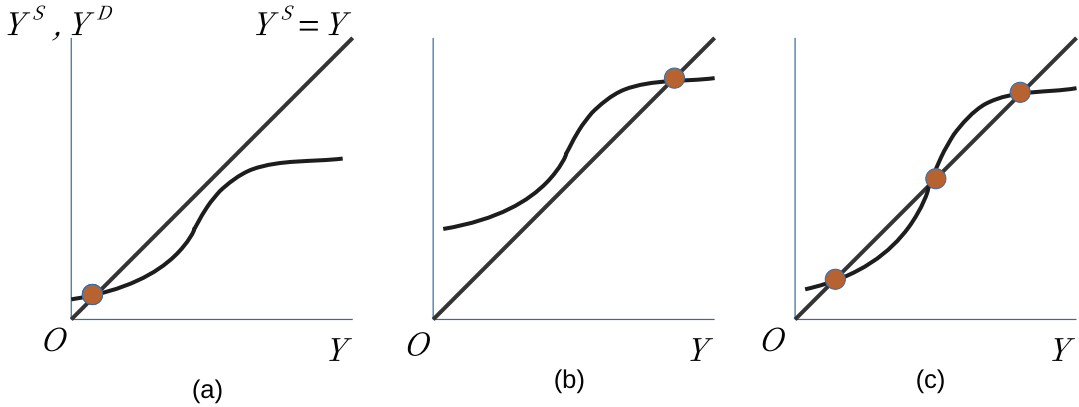
The Keynesian cross analysis is widely used as a theory of income determination. Based on the principle of effective demand developed by Keynes, this framework defines aggregate demand (Y^D), as the sum of consumption (C), investment (I), and government expenditure (G):

$$Y^D = C + I + G \tag{4}$$

Aggregate supply ultimately becomes income through its distribution in the form of profits, wages, or rents. Thus, the total value of aggregate supply is equivalent to national income:

$$Y^S = Y. \tag{5}$$

Figure 5: Keynesian cross analysis



In this section, we integrate the Keynesian cross analysis with the Kaldorian investment function. Kaldor (1940) presented a seminal contribution to business cycle theory by proposing an S-shaped investment function. According to his formulation, the responsiveness of investment to changes in national income is low at both the low and high levels of national income, while it is relatively high at intermediate levels of income.

During the recessionary phase, firms are constrained by substantial excess capacity, which discourages new investment activity; this implies that the derivative of investment with respect to income is small. Conversely, in the expansionary or boom phase, rising construction costs and limited access to external finance similarly inhibit investment, again leading to a low value of the derivative of investment with respect to income. Thus, the investment function exhibits a non-linear response to income, which plays a critical role in generating cyclical dynamics.

From the above discussion, three types of the Keynesian cross diagrams can be presented. Figure 5(a) shows that only a low-level equilibrium emerges. In this case, the low-level equilibrium is stable. Figure 5(b) indicates that only a high-level equilibrium appears. In this case, the unique high-level equilibrium is also stable. Finally, Figure 5(c) illustrates that all three equilibria appear simultaneously. In this case, both the low-level and high-level equilibria are stable, while the middle-level equilibrium is unstable. If the system deviates upwards from the middle-level equilibrium, it converges to the high-level equilibrium, and if it deviates downwards, it converges to the low-level equilibrium.

Turning now to a comparative static analysis, we investigate the properties of Keynesian equilibria. In both Figure 5(a) and Figure 5(b), expansive fiscal policies have the effect of raising the equilibrium level. In the case of Figure 5(c), while fiscal policies raise the levels of both the low-level and high-level equilibria, they have the effect of lowering the level of the middle-level equilibrium. This result is counterintuitive, as one might expect that expansive fiscal policies would uniformly increase equilibrium levels across all cases. However, as demonstrated, while fiscal expansion raises the low and high equilibria, it paradoxically deteriorates the middle-level equilibrium. This finding highlights the fact that fiscal policy can have different impacts depending on the existing equilibrium, revealing the importance of the current economic state in determining the policy's effectiveness.

4 A small open economy model

Extending the Keynesian cross analysis to an open economy setting where international trade is incorporated is a natural and straightforward progression. In this section, we introduce the international macroeconomic model that will be the subject of our analysis.

The aggregate demand function is given by:

$$Y^D = C + I + NX \quad (6)$$

where C represents consumption, I represents investment, and NX represents net exports or the trade balance, which is the difference between exports and imports. In this model, we disregard the role of the government sector for simplicity of analysis.

We now briefly describe each component of aggregate demand. The consumption function is given by

$$C = cY + C_0, \quad (7)$$

where $0 < c < 1$ is the marginal propensity to consume and $C_0 > 0$ is autonomous consumption. Investment is modeled as a function of output and capital:

$$I(Y, K) = f(Y) - \mu K, \quad (8)$$

where $\mu > 0$ captures the negative effect of capital stock on net investment. Exports are assumed to increase with foreign GDP, denoted by Y^F , while imports are positively related to domestic GDP, Y . Accordingly, the net export function is specified in linear form as:

$$NX = NX_0 + eY^F - mY \quad (9)$$

where $e > 0$ is the propensity to export, $m > 0$ is the propensity to import, and NX_0 is an autonomous component of net exports.

To analyze the dynamic behavior of the economy, we introduce two fundamental differential equations that capture the adjustment processes in the goods and capital markets. The first equation describes the short-run quantity adjustment mechanism in the goods market:

$$\dot{Y} = \alpha(C + I + NX - Y), \quad \alpha > 0 \quad (10)$$

where α denotes the speed at which output adjusts to excess demand in the goods market. The second equation represents the capital accumulation effect driven by investment:

$$\dot{K} = I - \delta K, \quad \delta > 0 \quad (11)$$

where $\delta > 0$ is the depreciation rate of capital.

At this stage, let us draw our attention on the treatment of the investment function, as its formulation could significantly affect the dynamic behavior and quantitative outcomes. In our numerical simulations, we use the following type of investment function:

$$f(x) = \frac{1}{a + b \exp[-(x - c)]} \quad (12)$$

To confirm that this function is an S-shaped curve, we shall check the derivatives of the function. First, we check the first derivative of this function:

$$f'(x) = \frac{b \exp[-(x - c)]}{\{a + b \exp[-(x - c)]\}^2} > 0. \quad (13)$$

This result shows that $f(x)$ is a monotonically increasing function of x . Furthermore, we can obtain the second derivative:

$$f''(x) = -\frac{b \exp[-(x - c)](a + b \exp[-(x - c)])(a - b \exp[-(x - c)])}{\{a + b \exp[-(x - c)]\}^4} \quad (14)$$

This result indicates that there is an inflection point on $f(x)$ at which the curvature changes sign. The inflection point is given by $x = x_I$ that satisfies $a = b \exp[-(x - c)]$. If $x = x_I$, $f''(x) = 0$. In addition, we can find that $f''(x) > 0$ for $x < x_I$ and $f''(x) < 0$ for $x > x_I$. It follows from the above results that $f(x)$ is an S-shaped curve.

In the following, the analysis will be divided into two cases. The first case considers the scenario where the foreign GDP (Y^F) is constant, while the second case addresses the situation where the foreign GDP is variable.

Case 1: Y^F is constant. In this case, we investigate the following system:

$$\dot{Y} = \alpha(cY + C_0 + f(Y) - \mu K + eY^W - mY + NX_0 - Y), \quad (15a)$$

$$\dot{K} = f(Y) - \mu K - \delta K. \quad (15b)$$

The steady-state values of the dynamic system, denoted by (K^*, Y^*) , are defined as follows:

$$cY^* + C_0 + f(Y^*) - \mu K^* + eY^W - mY^* + NX_0 - Y^* = 0 \quad (16a)$$

$$f(Y^*) - \mu K^* - \delta K^* = 0. \quad (16b)$$

To examine local stability around the steady state, we derive the Jacobian matrix evaluated at (K^*, Y^*) :

$$J_0 = \begin{bmatrix} \alpha(c + f'(Y^*) - m - 1) & -\alpha\mu \\ f'(Y^*) & -(\mu + \delta) \end{bmatrix}. \quad (17)$$

The characteristic equation associated with the Jacobian matrix is a second-order polynomial in λ :

$$\lambda^2 + b_1\lambda + b_2 = 0 \quad (18)$$

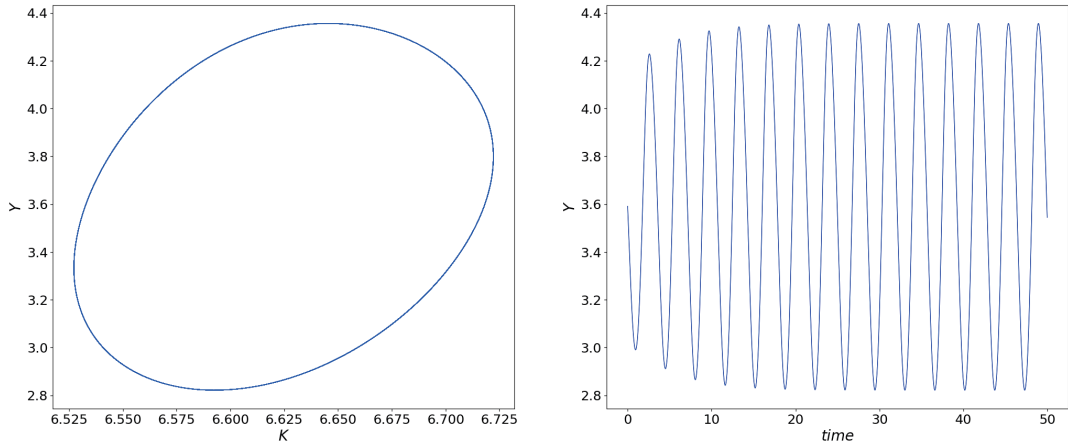
where

$$b_1 = -\alpha(c + f'(Y^*) - m - 1) + (\mu + \delta) \quad (19)$$

and

$$b_2 = -\alpha[(c + f'(Y^*) - m - 1)(\mu + \delta) - \mu f'(Y^*)] \quad (20)$$

Figure 6: Limit cycle



Stability of the steady state requires that the Routh-Hurwitz conditions be satisfied, i.e., $b_1 > 0$ and $b_2 > 0$. From these conditions, it follows that the steady state is stable if $c + f'(Y) - m - 1 < 0$.

It is worth emphasizing that, in Kaldor's (1940) model, the intermediate equilibrium is assumed to be unstable due to an excessive sensitivity of investment to output. This corresponds to the condition: $c + f'(Y) - m - 1 > 0$ at the intermediate equilibrium.

In the context of our model, if the inequality

$$(c + f'(Y^*) - m - 1)(\mu + \delta) - \mu f'(Y^*) < 0 \quad (21)$$

is satisfied, then the system undergoes a Hopf bifurcation, and the existence of a limit cycle can be demonstrated. On this point, see Mathematical Appendix 1. We will provide an accurate proof. In addition, a mathematical and theoretical discussion of Hopf bifurcations can be found in Guckenheimer and Holmes (1983) and Perko (2001).

Next, we proceed to the numerical simulations. Based on the above discussion, the investment function is specified as

$$I(Y, K) = \frac{1}{1 + 1.6 \exp[-(Y - 2.8)]} + 3.41 - 0.5K. \quad (22)$$

The net export function is assumed to be

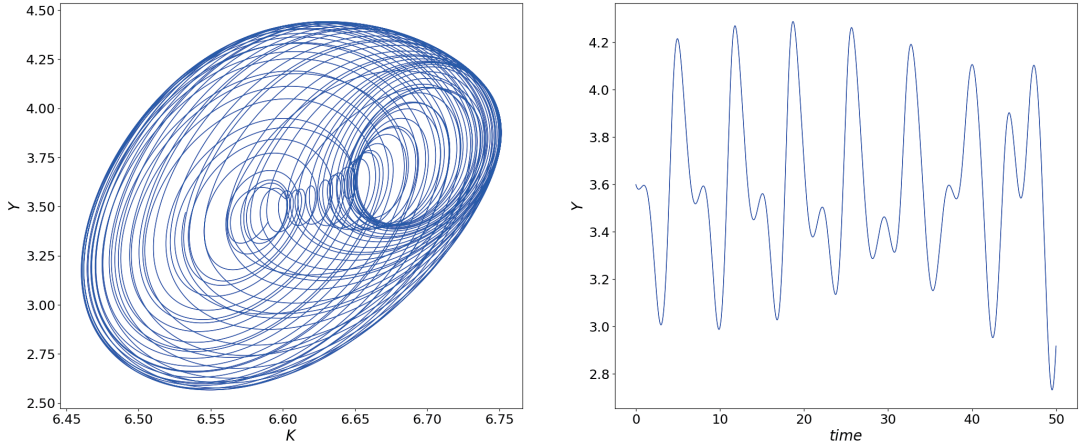
$$NX = 0.098 - 0.0175Y, \quad (23)$$

where foreign GDP is fixed. Furthermore, for other parameters, the following values are chosen:

$$\alpha = 29.5, c = 0.8, C_0 = 0.02, \delta = 0.1. \quad (24)$$

The left panel of Figure 6 depicts a limit cycle in the (K, Y) -plane, demonstrating the long-run behavior of the system. The cycle is both stable and evolves in a clockwise direction, indicating the nature of the dynamic interactions between the variables, K and Y . For ease of visualization, transient trajectories are omitted. The right panel of Figure 6 presents the time path of $Y(t)$ over the interval $0 \leq t \leq 50$. Starting from an initial condition inside the limit cycle, the trajectory gradually converges toward the cycle, confirming its stability.

Figure 7: Strange attractor



Case 2: Y^F is variable. In this case, we assume that

$$NX = NX_0 + f \cos(\theta t) - mY, \quad (25)$$

where the introduction of a cosine function implies that Y^F oscillates periodically with a constant amplitude. The external amplitude and driving frequency are f and m , respectively.

We examine the following system, which is a non-autonomous system of differential equations:

$$\dot{Y} = \alpha(cY + C_0 + f(Y) - \mu K + f \cos(\theta t) - mY + NX_0 - Y) \quad (26a)$$

$$\dot{K} = f(Y) - \mu K - \delta K. \quad (26b)$$

To perform numerical simulations, net export function is specified numerically as follows,

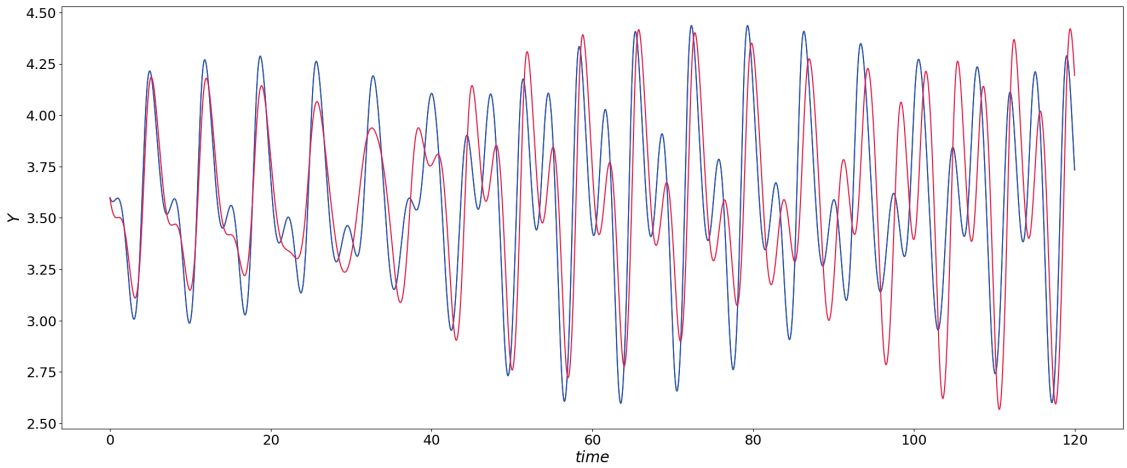
$$NX = 0.098 + 0.0265 \cos(.93t) - 0.0175Y. \quad (27)$$

Note that the external amplitude is small since $f = 0.0265$. The other functions and parameters remain the same as before.

In Figure 7, we plot the phase portrait and the the time path of the system with a weakly external oscillator. The left panel of Figure 7 depicts a strange attractor in the (K, Y) -plane, demonstrating the long-run behavior of the system. The right panel of Figure 7 provides the time path of $Y(t)$ over the interval $0 \leq t \leq 50$. Characteristic complex trajectories, indicative of nonlinear dynamical behavior, are evident in both panels.

The sensitive dependence on initial conditions is one of the key characteristics of chaos. In our model as well, this sensitive dependence on initial conditions was observed. We provided two slightly different initial values: $Y(0) = 3.6$, $K(0) = 6.7$ and $Y(0) = 3.6$, $K(0) = 6.71$. The results are shown in Figure 8. We can confirm that a small difference in the initial state of the system can lead to vastly different outcomes over time. Although two trajectories start close to one another in phase space, their future values can not be predicted. This phenomenon is often called as the butterfly effect, which means that a minor happening such as a butterfly flapping in Mexico can finally cause a serious change such as a tornado in Texas.

Figure 8: Sensitive dependence on initial conditions



5 Related Literature

Hans-Walter Lorenz (1987b) conjectured the possibility of chaotic dynamics in an international trade model involving three countries; however, he did not conduct any numerical simulations to support this claim. In contrast, Lorenz (1993) presented illustrative figures that suggest the presence of chaotic fluctuations within the same model framework.

In his book (1993) on nonlinear economic dynamics, he investigated an international trade model conceptualized as a system of coupled oscillators. He demonstrated that chaotic fluctuations can arise in a macroeconomic model involving three countries engaged in international trade. The countries are interconnected through trade relations, which act as coupling forces within the system. These coupling forces can be interpreted as perturbations of a motion on a three-dimensional torus. He presented the following model:

$$\dot{Y}_i = \alpha_i(C_i(Y_i, r_i) + I_i(Y_i, r_i) + EX_i(Y_j, Y_k) - IM(Y_i) - Y_i), \quad \alpha_i > 0, \quad (28a)$$

$$\dot{r}_i = \beta_i(L_i(Y_i, r_i) - M_i), \quad \beta_i > 0, \quad (28b)$$

$$\dot{M}_i = EX_i(Y_j, Y_k) - IM(Y_i), \quad (28c)$$

where $i, j, k = 1, 2, 3, j, k \neq i$.

Numerical simulations are a powerful method to investigate complex dynamics. In fact, he showed the occurrence of a strange attractor in the above model. From a mathematical point of view, his analysis was an application of Newhouse, Ruelle and Takens (1978). They demonstrated that a small perturbation on a three-dimensional torus could produce a strange attractor. Lorenz's model offered an insightful example of how complex dynamics – such as chaos – can emerge from economic interactions via international trade. In addition, Lorenz (1987a) showed the occurrence of strange attractors in a multi-sector model.

Finally, let us briefly discuss the differences between our model and the Lorenz model. The Lorenz model is based on the IS-LM framework, whereas our model is grounded in the Keynesian cross analysis. Accordingly, in the Lorenz model, both GDP and the interest rate are endogenous variables. Furthermore, since it assumes that trade imbalances affect the money supply of each country, the money supply is also treated as an endogenous

variable. Given that the model encompasses three countries, the Lorenz model constitutes a high-dimensional system with nine endogenous variables. In contrast, our model consists of only two variables — GDP and capital stock — making it a low-dimensional system. However, as is well known, chaos does not arise in two-dimensional continuous-time dynamical systems; the most complex behavior possible is at most a limit cycle. At first glance, this appears logically inconsistent. Nevertheless, the emergence of chaotic behavior in our model can be attributed to its nature as a non-autonomous system with external forcing. In this sense, our model may be regarded as an application of the Ueda model.

6 Concluding remarks

This paper investigated the emergence of business cycles in a small open economy by developing a dynamic macroeconomic model that exhibits both limit cycles and complex nonlinear behaviors. The analytical framework built upon the Keynesian cross model, augmented with a Kaldorian (1940) investment function that explicitly depends on the level of output (GDP) and the existing capital stock. This formulation of investment was pivotal to the model's explanatory power to generate endogenous cyclical dynamics. In addition, international trade was incorporated through a net exports function, which served as a key driver of aggregate demand and contributed further to the propagation of economic fluctuations.

The analysis was conducted under two distinct scenarios: the case of constant foreign GDP and the case of periodically varying foreign GDP. In the first scenario, we assumed that foreign GDP remained fixed over time. Under this condition, we established the existence of limit cycles by applying the Hopf bifurcation theorem, thereby providing a rigorous theoretical basis for the emergence of self-sustained oscillations in output. Numerical simulations were employed to validate the theoretical predictions, confirming the presence of a stable limit cycle and offering quantitative insight into the dynamics of the model.

In the second scenario, we relaxed the assumption of constant foreign output and instead introduce periodic fluctuations, modeled as a cosine function of time. This resulted in a non-autonomous system of differential equations, complicating the qualitative analysis. Consequently, we relied on numerical methods to explore the behavior of the system. Simulation results revealed the emergence of a strange attractor under certain parameter configurations, suggesting the potential for chaotic dynamics driven by externally induced economic variability.

This study offered a comprehensive examination of business cycle phenomena in a small open economy context. It demonstrated that both stable periodic cycles and chaotic fluctuations might arise endogenously, depending on the nature of external economic conditions. By combining analytical and numerical techniques, the paper contributed to a deeper understanding of the role of investment dynamics and international trade in shaping macroeconomic instability.

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Mathematical Appendix 1: Hopf bifurcation theorem

Hopf bifurcation theorem is a useful and effective tool to detect the existence of closed orbits for the system of ordinary differential equations. Since our system currently under consideration is the two-dimensional system, we will introduce the following two-dimensional version:

(Hopf Bifurcation Theorem) Consider the system of ordinary differential equations on an open set $U \in \mathbb{R}^2$, $\dot{x} = f(x, \mu)$, where $x \in U$ and μ is a real parameter varying in some open interval $I \in \mathbb{R}$.

Suppose that for each μ in I there exists an equilibrium point $x^* = x^*(\mu)$ of the above system. Assume that the Jacobian matrix of f with respect to x , evaluated at $x^*(\mu)$, has a pair of complex conjugate eigenvalues, $\lambda(\mu)$ and $\bar{\lambda}(\mu)$, which satisfy the following conditions:

$$(H1) \operatorname{Re} \lambda(\mu_H) = 0, \operatorname{Im} \lambda(\mu_H) \neq 0,$$

$$(H2) \left. \frac{d \operatorname{Re} \lambda(\mu)}{d \mu} \right|_{\mu=\mu_H} \neq 0$$

Then, the dynamical system has a family of non-constant, periodic solutions.

By applying the Hopf bifurcation theorem, we can show that a limit cycle appears in the international trade model in Case 1. In this case, the bifurcation parameter is chosen to be α . It is easy to check that both (H1) and (H2) are satisfied at $\alpha_H = (\mu + \delta)/(c + f'(Y^*) - m - 1)$.