Bayesian Estimation of a Stable Distribution Using the Hamiltonian Monte Carlo Method with Application to Stock Indices

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1. Introduction

Stable distributions have two characteristics – skewness and fat tails – that make them particularly applicable to a wide range of areas in economics, physics, and biology[1)]. Estimating the parameters of the probability density function for a stable distribution typically uses the scaling unbiasedness of the probability density function[2)], assuming ergodicity[3)], and Bayesian inference that approximates the stable distribution via a Poisson series expansion[4)]. However, Bayesian inference based on the Markov chain Monte Carlo (MCMC) method has been shown to perform satisfactorily. In this study, parameter estimation of the probability density function for the stable distribution is performed by Bayesian inference using the Hamiltonian Monte Carlo (HMC) method[5)].

In the empirical analysis conducted in this study, we use daily data for the Nikkei Stock Average (Nikkei), the Tokyo Stock Price Index (TOPIX), the Dow Jones Industrial Average (DJIA), and the Standard & Poor's 500 Stock Index (S&P500). Based on our estimates of the stability parameter of the probability density function for the stable distribution, our analysis showed that the distributions of the daily returns for the Nikkei, TOPIX, DJIA, and S&P500 have thick tails. Based on the results of our skewness parameter estimation, we found that the Nikkei and TOPIX have a distorted distribution, while the DJIA and S&P 500 have an undistorted distribution. Importantly, we found that the differences in the distributions of the daily returns are derived from differences in the scale parameter. The relatively small inefficiency factor (IF) values for the Bayesian estimation of the probability density function for the stable distribution conducted here support the effectiveness of the HMC method.

The remainder of this paper is organized as follows: Section 2 describes the probability density function for the stable distribution and the numerical integration used in this study. Section 3 explains Bayesian inference with the HMC method using Leapfrog. Section 4 describes the data and the empirical results for the four stock indices used in the analysis and discusses their implications. Section 5 presents conclusions and identifies future challenges.

2. Stable Distribution

2.1 Stable Distribution

When X_1 and X_2 are mutually independent replications of a random variable X, X is said to be stable if

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$aX_1 + aX_2$ for any constants $a > 0$ and $b > 0$ follows the distribution $cX + d$ for some constants $c > 0$ and d. Distribution functions with this property are generally called "stable distributions." Functions such as a normal distribution, Cauchy distribution, and Lévy distribution, satisfy the above property and belong to a special class of stable distributions. The probability density function $f(x|\alpha,\beta,\gamma,\delta)$ for a stable distribution cannot be obtained analytically[6)], but is defined using the Fourier transform of the characteristic function $\psi(t|\alpha,\beta,\gamma,\delta)$ as follows:

$$f(x|\alpha,\beta,\gamma,\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(t|\alpha,\beta,\gamma,\delta)e^{-ixt}dt. \tag{2.1}$$

Here, the characteristic function is given by

$$\psi(t|\alpha,\beta,\gamma,\delta) = \exp[it\delta - |\gamma t|^{\alpha}(1 - i\beta\,\mathrm{sgn}\,(t)\boldsymbol{\Phi}(t,\alpha,\gamma))]. \tag{2.2}$$

Here,

$$\boldsymbol{\Phi}(t,\alpha,\gamma) = \begin{cases} (|\gamma t|^{1-\alpha} - 1)\tan\left(\dfrac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ -\dfrac{2}{\pi}\log|\gamma t| & \alpha = 1 \end{cases}. \tag{2.3}$$

A stable distribution is characterized by four parameters of the characteristic function $(\alpha,\beta,\gamma,\delta)$. α is the stability parameter, which takes a value in the range $0 < \alpha \leq 2$, indicating the thickness of the base of the stability distribution. β is the skewness parameter; it has a value in the range $-1 \leq \beta \leq 1$ and indicates the symmetry of the stable distribution. γ is the scale parameter and represents the variability of the stable distribution. δ is the location parameter, which serves to translate the stable distribution. There is no general analytical solution for Eq. (2.1), but the combination of the stability parameter and the skewness index allows us to express the stable distribution as an elementary function as a special case. When the stability parameter $\alpha = 2$, the stable distribution is normal with a mean of δ and a variance of $\sigma^2 = 2\gamma$. When the stability parameter $\alpha = 1$ and the skewness parameter $\beta = 0$, the distribution is a Cauchy distribution with a location parameter δ and a scale parameter γ. For a stability parameter of $\alpha = 0.5$ and a skewness parameter of $\beta = 1$, the distribution is a Lèvy distribution with a location parameter δ and a scale parameter γ.

If the observed data is $\boldsymbol{y} = \{y_i\}_{i=1}^{N}$, the likelihood function for the probability density function for the stable distribution is expressed as follows:

$$\begin{aligned} L(\boldsymbol{y}|\boldsymbol{\theta}) &= \prod_{i=1}^{N} f(y_i|\alpha,\beta,\gamma,\delta) \\ &= \prod_{i=1}^{N}\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(t|\alpha,\beta,\gamma,\delta)e^{-iy_it}dt. \end{aligned} \tag{2.4}$$

Here, $\boldsymbol{\theta} = (\alpha,\beta,\gamma,\delta)$.

2.2 Numerical Integration of Stable Distribution

It is necessary to perform the numerical product of the Fourier integrals with high accuracy in order to obtain the probability density function for a stable distribution. Ament and O'Neil (2018) divided the cases according to the presence or absence of distortion ($\beta = 0$ or $\beta \neq 0$) and performed asymptotic expansions for x and α, using the generalized Gaussian quadrature formula[7)]. However, this method is not effective for high oscillatory integrals or oscillatory integrals with slow decay, and there is a problem that the density function for the stable distribution cannot be calculated for the range of parameters that requires such calculations. Therefore, for the Fourier integral of the density function for the stable distribution, we use the double exponential formula (DE)[8], which is an effective method for oscillatory integrals. For Fourier integration of an integrand with slow decay as $x \to \infty$,

$$F(\omega)=\int_0^{\infty} f(x)e^{i\omega x}dt, \tag{2.5}$$

If we perform a variable transformation $x = M\phi(t)$ and

$$\phi(t)=\frac{t}{1-\exp\{-2t-a(1-e^{-t})-b(e^{t}-1)\}}, \tag{2.6}$$

we have

$$F(\omega)=\int_{-\infty}^{\infty} f(M\varphi(t))\exp(i\omega M\varphi(t))\varphi'(t)dt. \tag{2.7}$$

Next,

$$E(\omega)=\int_{-\infty}^{\infty} f(M\varphi(t))\exp\left(i\omega M\varphi(t)-\frac{i}{2}\omega_0 M\hat{\varphi}(t)\right)\varphi'(t)dt \tag{2.8}$$

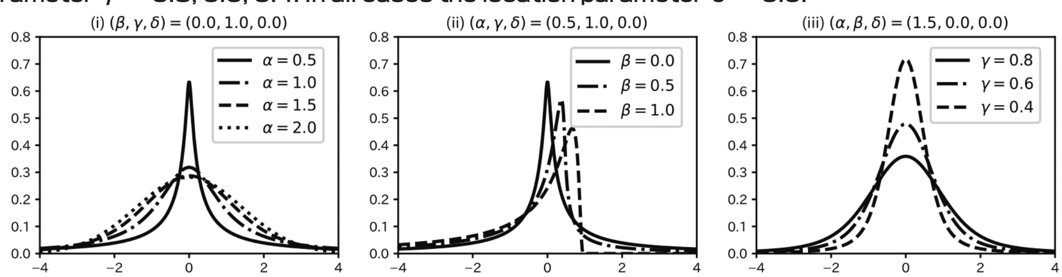
is defined. Here, $\hat{\varphi}(t)=\varphi(t)-t$ and ω_0 is a positive constant. The function $|E(\omega)|$ is very small for large M. The DE formula $\tilde{F}(\omega)$ for Eq. (2.5) is given by $\tilde{F}(\omega)=F(a)-E(\omega)$,

$$\tilde{F}(x)=\int_{-\infty}^{\infty} f(M\varphi(t))\exp\left(i\omega M\varphi(t)-\frac{i}{2}\omega_0 M\hat{\varphi}(t)\right) \times 2iM\sin\left(\frac{1}{2}\omega_0 M\hat{\varphi}(t)\right)\varphi'(t)dt \tag{2.9}$$

is obtained. While the range of integration of the DE formula for the Fourier integral is $t \geq 0$, the range of integration for the stable distribution in Eq. (2.1) is $(-\infty,\infty)$. However, from the nature of the characteristic function for the stable distribution in Eq. (2.2), we can always arrive at the calculation of $t \geq 0$, and the condition $x > 0$ is satisfied except for $x = 0$.

Figure 1 shows the stability distribution drawn using several parameters to examine the relationship between the

Figure 1: Examples of the probability density distribution of the stable distribution. (i) Case with the skewness parameter β fixed at 0, the scale parameter γ fixed at 1.0, and stability parameters α = 0.5, 1.0, 1.5, 2.0, (ii) Case with the stability parameter α fixed at 0.5, the scale parameter γ fixed at 1.0, and the skewness parameter β = 0.0, 0.5, 1.0, (iii) Case with the stability parameter α fixed at 1.5 and the skewness parameter β fixed at 0.0, and the scale parameter γ = 0.8, 0.6, 0.4. In all cases the location parameter δ = 0.0.



parameters and the asymmetry and thickness of the base. Figure 1(i) shows the stable distributions for different values of the characteristic index $\alpha=0.5$, 1.0, 1.5, 2.0, with the skewness index $\beta=0$ and the scale parameter $\gamma=1.0$ fixed. As the characteristic index increases, the base of the distribution becomes thicker, and the distribution becomes a normal distribution at $\alpha=2$. Figure 1(ii) shows the stable distributions when the characteristic index α is fixed at 0.5 and the scale parameter γ is fixed at 1.0, and the skewness index β is changed to 0.0, 0.5, and 1.0. The reason for choosing the characteristic index $\alpha=0.5$ is that it is a parameter value for which the calculation of the probability density function has been abandoned by Ament and O'Neil (2018). As the skewness index increases, the distortion of the distribution becomes larger. At $\beta=1.0$, the distribution is consistent with a Lévy distribution. Figure 1(iii) shows the stable distribution when the characteristic index $\alpha=1.5$, the skewness index $\beta=0$, and the scale index $\gamma=0.8$, 0.6, and 0.4 are varied. The scale index is a parameter corresponding to the standard deviation in a normal distribution; the smaller the scale index, the smaller the variability. Figure 1 confirms the validity of the DE formula. From Figure 1, we can confirm the validity of the DE formula.

3. HMC Methohd

3.1 Hamiltonian Dynamics

In the HMC method, the molecular dynamics method developed in the field of physics is used to update the parameters. The molecular dynamics method is a simulation method that reproduces the motion of atoms and molecules by the numerical integration of equations of motion based on Hamiltonian dynamics. In this section, we briefly describe Hamiltonian dynamics, which is necessary to explain the HMC method. In Hamiltonian dynamics, the state of a system is defined as a phase space with generalized coordinates and momenta $(q_j,p_j),(j=1,\dots,d)$, and the time evolution of the system is given by a trajectory in the phase space.

The equation for the trajectory representing the time evolution of the system is Hamilton's equation of motion:

$$\frac{dq_j}{d\tau}=\frac{\partial H}{\partial p_j}, \tag{3.1}$$

$$\frac{dp_j}{d\tau}=-\frac{\partial H}{\partial q_j}. \tag{3.2}$$

Here, τ is the time. H is called the Hamiltonian and represents the energy of the system, and

$$H(q,p)=\sum_{j=1}^{d}\frac{p_j^2}{2m_j}+U(q)\equiv K(p)+U(q) \tag{3.3}$$

is defined. $K(p)$ and $U(q)$ are the kinetic and potential energies of the system, respectively; m_j is the mass of the point mass.

3.2 Bayesian Inference and HMC Metohd

Assuming that the parameters of the statistical model are $\boldsymbol{\theta}=\{\theta_j\}_{j=1}^{d}$ and the data are $\boldsymbol{y}=\{y_i\}_{i=1}^{n}$, the posterior distribution $f(\boldsymbol{\theta}|\boldsymbol{y})$ in the Bayesian estimation method is expressed asfollows:

$$f(\boldsymbol{\theta}|\boldsymbol{y})=\frac{f(\boldsymbol{y}|\boldsymbol{\theta})f(\boldsymbol{\theta})}{\int f(\boldsymbol{y}|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}}. \tag{3.4}$$

Here, $f(\boldsymbol{y}|\boldsymbol{\theta})$ is the likelihood function, and $f(\boldsymbol{\theta})$ is the prior distribution of the parameter $\boldsymbol{\theta}$. The denominator in

Eq. (3.4) is a normalization factor and is also called the marginal likelihood. In this paper,in order to reduce the influence of the prior distribution on the posterior distribution, the non-information prior distribution is used as the prior distribution. The non-information prior distribution is the distribution used when there is no information about the parameters in advance or when there is no basis for setting the prior distribution.

We introduce a standard normal distribution $f(\boldsymbol{p})$ that is independent of the posterior distribution in Eq. (3.4) and $\boldsymbol{p}=\{p_j\}_{j=1}^{d}$, called the conjugate momentum of the parameter $\boldsymbol{\theta}$. The standard normal distribution of this conjugate is

$$f(\boldsymbol{p})=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\boldsymbol{p}^2}{2}\right) \tag{3.5}$$

and the joint distribution $f(\boldsymbol{\theta},\boldsymbol{p}\,|\,\boldsymbol{y})$ with the joint distribution in Eq. (3.4) is given by

$$f(\boldsymbol{\theta},\boldsymbol{p}\,|\,\boldsymbol{y})=f(\boldsymbol{\theta}\,|\,\boldsymbol{y})f(\boldsymbol{p}). \tag{3.6}$$

In the HMC method, random numbers are generated from this joint distribution. From Eq. (3.5), this becomes

$$\begin{aligned} f(\boldsymbol{\theta},\boldsymbol{p}\,|\,\boldsymbol{y}) &\propto \exp\left(-\frac{\boldsymbol{p}^2}{2}+\log f(\boldsymbol{y}\,|\,\boldsymbol{\theta})\right) \\ &=\exp\left[-H(\boldsymbol{\theta},\boldsymbol{p})\right]. \end{aligned} \tag{3.7}$$

Here,

$$H(\boldsymbol{\theta},\boldsymbol{p})=\frac{1}{2}\boldsymbol{p}^2-\log f(\boldsymbol{y}\,|\,\boldsymbol{\theta}) \tag{3.8}$$

is defined as the Hamiltonian. In the HMC method, the parameters $\boldsymbol{\theta}$ and the conjugate momentum $\boldsymbol{p}$ are assumed to follow Eqs. (3.1) and (3.2). However, τ here represents the time virtually introduced. The generation of the random number sequence $\boldsymbol{\theta}^{(k)}(k=0,1,2,...)$ by the HMC method is executed by following the six Monte Carlo steps listed below:

1. Generate the conjugate momentum $p_i^{(k)}$ of the parameter $\theta_i^{(k)}$ as a random number following the standard normal distribution $\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(p_i^{(k)})^2}{2}\right]$. Note that the random generation of the conjugate momentum $p_i^{(k)}$ is a Markov chain.
2. Compute the Hamiltonian $H_{init}=H(\boldsymbol{\theta}^{(k)},\boldsymbol{p}^{(k)})$ at the initial time $\tau=0$.
3. By numerically solving the differential equations (3.1) and (3.2) using the Leapfrog integrals described in Section 3.3, we can develop the time evolution from the initial state $(\theta_i^{(k)}(\tau=0),p_i^{(k)}(\tau=0))=(\theta_i^{(k)},p_i^{(k)}(\tau=\tau_{fin}))$ to the terminal condition $(\theta_i^{(k)}(\tau=\tau_{fin}),p_i^{(k)}(\tau=\tau_{fin}))$.
4. Calculate the Hamiltonian $H_{fin}=H(\boldsymbol{\theta}^{(k)}(\tau=\tau_{fin}),\boldsymbol{p}^{(k)}(\tau=\tau_{fin}))$ for the terminal condition.
5. Select a new candidate random number with probability r.

$$r=\min\left[1,\exp\left(-H_{fin}+H_{init}\right)\right]. \tag{3.9}$$

6. Repeat the steps above.

3.3 Leapfrog Integration

Since Hamilton's equation of motion in Eqs. (3.1) and (3.2) cannot be solved analytically, it is necessary to differentiate and solve numerically. To meet the detailed balance conditions for the HMC method, the numerical integration method in Hamilton's equation of motion in Eqs. (3.1) and (3.2) must satisfy the conditions of time inversion and volume conservation. Leapfrog integration uses a numerical integral that meets these conditions. In Leapfrog integration, the parameter $\boldsymbol{\theta}$ and conjugate momentum $\boldsymbol{p}$ are time-evolved. That is, the parameter $\boldsymbol{\theta}$ and the conjugate momentum $\boldsymbol{p}$ are defined by the time offset by half the step size ϵ. Equations (3.1) and (3.2) differentiated by the Leapfrog integral are as follows:

$$p_j^{(k)}\left(\tau+\frac{\epsilon}{2}\right)=p_j^{(k)}(\tau)-\left(\frac{\epsilon}{2}\right)\frac{\partial H(\boldsymbol{\theta},\boldsymbol{p})}{\partial \theta_j}\bigg|_{\theta=\theta(\tau)}, \tag{3.10}$$

$$\theta_j^{(k)}(\tau+\epsilon)=\theta_j^{(k)}(\tau)+\epsilon p_j^{(k)}\left(\tau+\frac{\epsilon}{2}\right), \tag{3.11}$$

$$p_j^{(k)}(\tau+\epsilon)=p_j^{(k)}\left(\tau+\frac{\epsilon}{2}\right)-\left(\frac{\epsilon}{2}\right)\frac{\partial H(\boldsymbol{\theta},\boldsymbol{p})}{\partial \theta_j}\bigg|_{\theta=\theta(\tau+\epsilon)}. \tag{3.12}$$

From the virtual time τ_{fin} that advances in one Monte Carlo step and the step size ϵ of the Leapfrog integral, the number of steps N that repeat the Leapfrog integral satisfies $\tau_{fin}=\epsilon N$.

3.4 Calculation Conditions and Convergence Diagnosis

In this study, to conduct our Bayesian estimation with the HMC method, we discarded the first 1,000 samples as the burn-in period, then generated and extracted 10,000 probability samples to guarantee the independence of the MCMC samples. The interval was set to 2, and 5,000 samples were used. The number of MCMC chains running simultaneously was set to 3. All of the numerical calculations were performed using the C language.

To validate the HMC method, it is necessary to judge whether the value obtained by sampling converges to a posterior distribution. There are several methods for diagnosing this convergence. In this paper, the judgment is based on both a visual time-series plot of the generated probability sample and Gelman-Rubin (G-R) statistics[9)]. A G-R statistic close to 1 is interpreted to mean that the chain converges to a steady state. Specifically, if the G-R statistic is less than 1.05, it is judged that the parameter converges to a stable state.

4. Empirical Analysis

4.1 Data

In our empirical analysis, we used the closing prices of the Nikkei, TOPIX, DJIA, and S&P500 obtained from Bloomberg. If S_t is the stock price at t, the daily rate of return y_t at t is calculated as $y_t=(\ln S_t-\ln S_{t-1})\times 100$. The data period for the Nikkei and TOPIX is January 5, 2015, to December 30, 2019. The sample period for each daily return is January 6, 2015, to December 30, 2019, giving us a total of 1,221 observations. The data period for the DJIA and S&P500 is January 2, 2015, to December 31, 2019, and the sample period for each daily return is January 3, 2015, to December 31, 2019; the number of observations is 1,257. Table 1 shows the summarized statistics for the daily returns for each stock index. Figure 2 shows the histograms of the daily returns. As can be seen, the Nikkei and TOPIX histograms have thicker tails and a lower distribution around the mean than is the case for the DJIA and S&P500.

4.2 Empirical Results

Figures 3 through 6 show the sample autocorrelation function, sample path, and posterior probability density

Table 1: Summary of statistics for daily returns

Nikkei, TOPIX: January 6, 2015 – December 30, 2019
DJIA, S&P500: January 3, 2015 – December 31, 2019

Stock Index	No. of Obs.	Mean	Std. Dev.	Skewness	Exc. Kurtosis	Max.	Min.
Nikkei	1,221	0.0251	1.2257	−0.3414	6.1453	7.4262	−8.2529
TOPIX	1,221	0.0169	1.3606	−0.3498	6.6610	7.7153	−7.5324
DJIA	1,257	0.0374	0.8541	−0.5350	3.8342	4.8643	−4.7143
S&P500	1,257	0.0359	0.8476	−0.5248	3.8366	4.8403	−4.1843

Figure 2: Histograms of daily returns for the Nikkei, TOPIX, DJIA and S&P500.

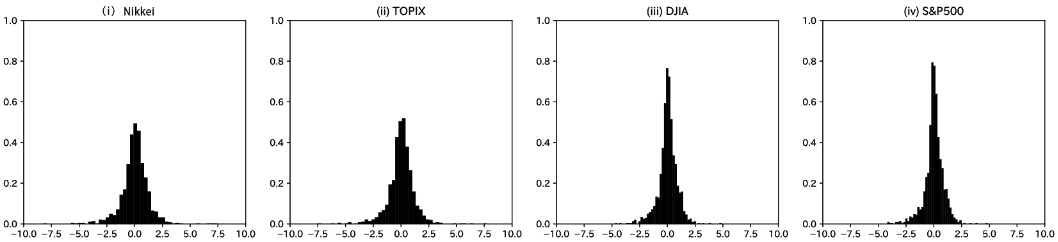
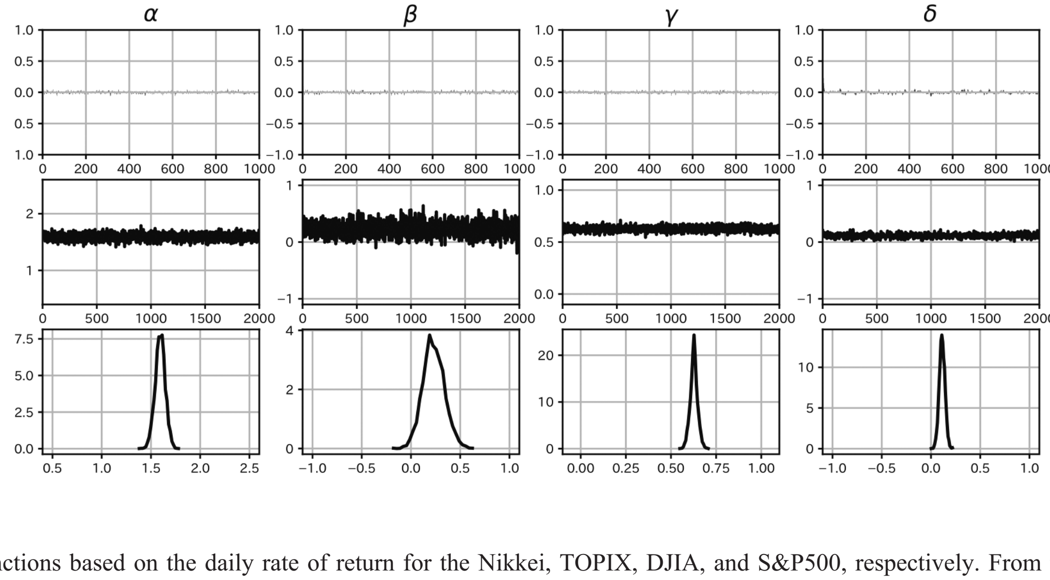


Figure 3: Estimation results for the daily returns for the Nikkei, sample autocorrelation function (upper), sample path (middle), posterior probability density function (lower).



functions based on the daily rate of return for the Nikkei, TOPIX, DJIA, and S&P500, respectively. From these figures, it appears that the sample autocorrelation function is sufficiently attenuated for all parameters and quickly converges to a stationary distribution. In addition, the sample path visits the state space sufficiently and the posterior probability density function has a monomodal distribution. Table 2 shows the G-R statistics for the four parameters of the probability density function for the stable distributions of the daily rate of return for the Nikkei, TOPIX, DJIA, and S&P500. As shown, the G-R statistic is 1.05 or less for the four parameters of the probability

Figure 4: Estimation results for the daily returns for the TOPIX, sample autocorrelation function (upper), sample path (middle), posterior probability density function (lower).

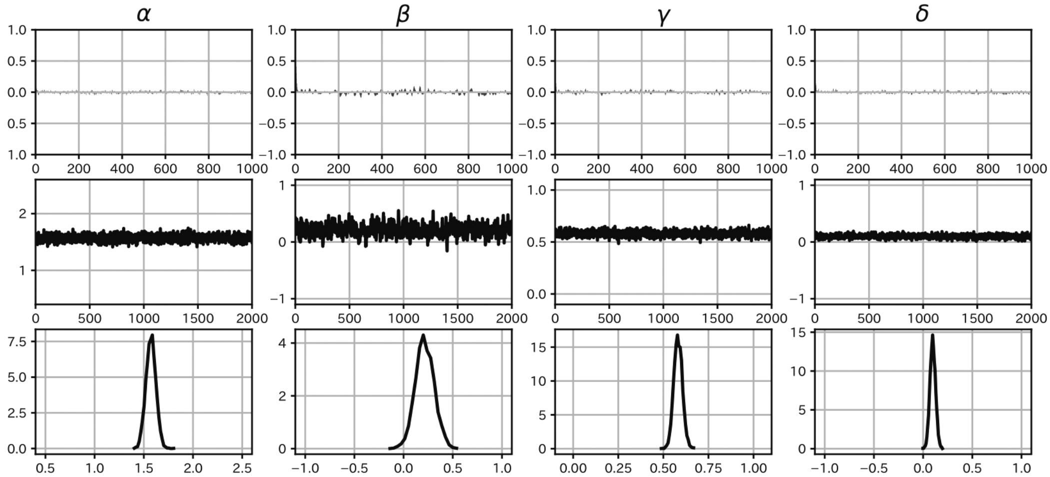
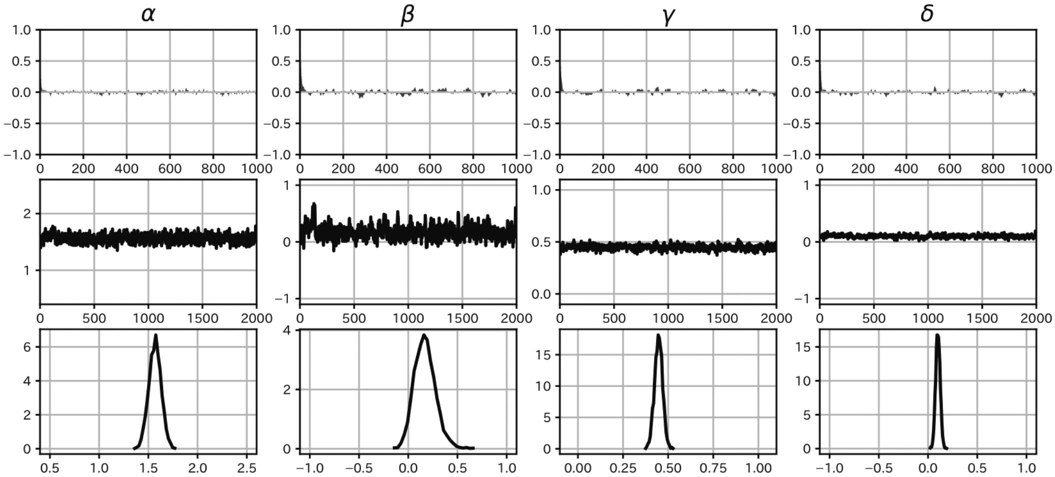


Figure 5: Estimation results for the daily returns for the DJIA, sample autocorrelation function (upper), sample path (middle), posterior probability density function (lower).



density function for the stable distribution based on the daily rates of return for the four indices. The results in Figures 1 through 6 and the G-R statistics in Table 2 indicate that the sample series obtained using the Nikkei, TOPIX, DJIA, and S&P500 sufficiently converge to an invariant distribution. Table 3 shows the estimated probability density function parameters of the stable distributions.

As can be seen, the posterior mean of the stability parameter based on the daily returns for the Nikkei is 1.5930, with a 95% credible interval of [1.4922, 1.6915]. That the interval boundary values are both less than 2.0 indicates that the distribution is thicker than a normal distribution. This result is consistent with previous studies such as those by Fukunaga and Umeno (2018) and Bielinsky *et al.* (2019). The 95% credible interval for the skewness parameter is [0.0249, 0.4446]. Since the interval does not contain zero, the indication is that there is an asymmetry

Figure 6: Estimation results for the daily returns for the S&P500, sample autocorrelation function (upper), sample path (middle), posterior probability density function (lower).

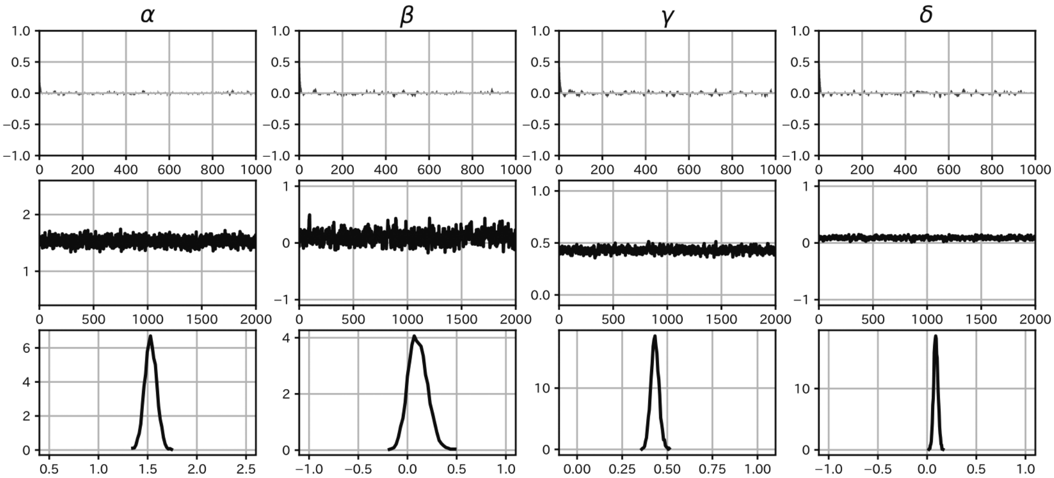


Table 2: Gelman-Rubin statistics

	α (stability)	β (skewness)	γ (scale)	δ (location)
Nikkei	1.0000	1.0000	1.0000	1.0002
TOPIX	1.0002	1.0006	1.0000	1.0001
DJIA	1.0002	1.0007	1.0001	1.0003
S&P500	1.0001	1.0011	1.0012	1.0005

Table 3: Estimation results for parameters

Line 1: Posterior Mean (Std. Dev.), Line 2: 95% Credible Interval, Line 3: IF

	α (stability)	β (skewness)	γ (scale)	δ (location)
Nikkei	1.5930 (0.0510)	0.2288 (0.1074)	0.6269 (0.0208)	0.1134 (0.0291)
	[1.4922, 1.6915]	[0.0249, 0.4436]	[0.5850, 0.6691]	[0.0558, 0.1706]
	2.92	3.09	3.07	6.53
TOPIX	1.5717 (0.0507)	0.2176 (0.0971)	0.5818 (0.0237)	0.0985 (0.0285)
	[1.4729, 1.6695]	[0.0312, 0.4113]	[0.5361, 0.6290]	[0.0439, 0.1544]
	4.42	5.48	5.61	3.96
DJIA	1.5659 (0.0625)	0.1648 (0.1067)	0.4468 (0.0226)	0.0996 (0.0227)
	[1.4448, 1.6874]	[−0.0273, 0.3942]	[0.4031, 0.4909]	[0.0563, 0.1447]
	8.21	11.42	8.70	8.90
S&P500	1.5306 (0.0624)	0.1039 (0.0993)	0.4312 (0.0227)	0.0890 (0.0223)
	[1.4090, 1.6527]	[−0.0791, 0.3082]	[0.3867, 0.4756]	[0.0468, 0.1346]
	4.92	8.67	6.95	6.32

in the distribution. The posterior mean of the scale index is 0.6269, which is the largest scale index value among the four stock indices. The IF values shown in the table represent the number of samples needed to make the standard error of the sample mean equal to the standard error that would result from random sampling. The single-digit IF values for all four parameters indicate that the HMC method is quite efficient.

The estimation results using the daily returns for the TOPIX are similar to those for the Nikkei. As shown in Table 3, the posterior mean of the stability parameter and the 95% credible interval are 1.5717 and [1.4729, 1.6695], respectively. As in the case of the Nikkei, since the interval boundaries are both smaller than 2, the distribution of the TOPIX daily returns is thicker than a normal distribution. Moreover, since the 95% credible interval for the skewness parameter is [0.0312, 0.4113] and thus does not include zero, there is an asymmetry in the distribution, as was true for the Nikkei.

For the DJIA daily returns, the posterior mean of the stability parameter is 1.5659, with a 95% credible interval of [1.4448, 1.6874]. These values are nearly the same as the corresponding values for the Nikkei and the TOPIX. Given that the interval boundaries are both less than 2, the distribution is thicker than the normal distribution. The 95% credible interval for the skewness parameter, [−0.0273, 0.3942], includes zero, meaning that the DJIA distribution of daily returns has an asymmetry unlike the Nikkei and TOPIX. The posterior mean of the scale index is 0.4468, which is smaller than the corresponding values for the Nikkei and TOPIX.

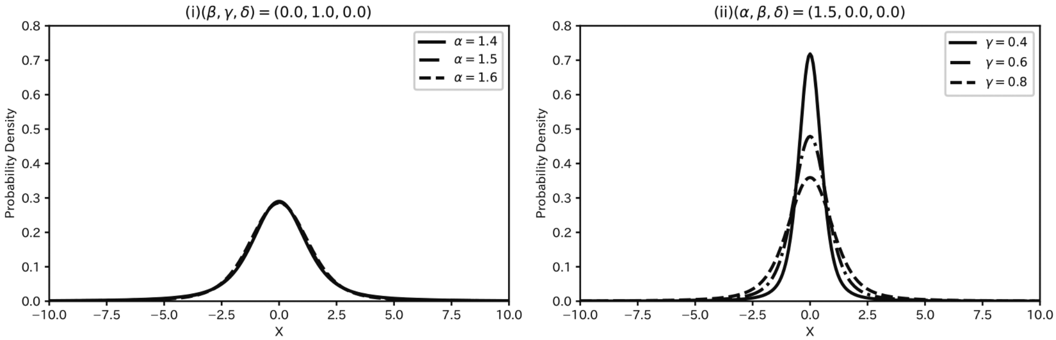
In the case of the S& P500, the posterior mean of the stability parameter and the 95% credible interval are 1.5306 and [1.4090, 1.6527], respectively. These values are very similar to the values obtained for the Nikkei, TOPIX, and DJIA. Once again, the interval boundaries are both less than 2, indicating that the distribution of daily returns is thicker than a normal distribution. Since the 95% credible interval for the skewness parameter is [−0.0791, 0.3082], and thus contains zero, there is no asymmetry, as was the case for the DJIA. The posterior mean of the scale parameter is 0.4312, the smallest value among the four stock indexes.

4.3 Discussion

As indicated by the distributions in Figure 2, the shapes of the distributions of daily returns in Japan and the United States are different. However, the estimated stability parameter, which characterizes the shape of the distribution estimated from the parameter estimation of the stable distribution, is similar for all four stock indices. Thus, we can reasonably conclude that the differences in the daily return distributions are not derived from differences in the stability parameter. This fact shifts the focus to the differences in the scale parameter. The posterior means of the scale parameter γ for the Nikkei, TOPIX, DJIA, and S&P500 are 0.6269, 0.5818, 0.4468, and 0.4312, respectively; the corresponding 95% credible intervals are [0.5850, 0.6691], [0.5361, 0.6290], [0.4031, 0.4909], and [0.3867, 0.4756]. To illustrate the relative influence of the stability parameter and the scale parameter of the probability density function for the stable distribution, Figure 7 shows a graph of the probability density function for the stable distribution when the stability parameter and the scale parameter are changed.

From Figure 7(i), it can be seen that the shape of the probability density function for the stable distribution does not change significantly when the scale parameter is fixed and the stability parameter is changed to $\alpha=1.4, 1.5, 1.6$. In contrast, it can be seen in Figure 7(ii) that the shape of the probability density function for the stable distribution changes noticeably when the stability parameter is fixed and the scale parameter is changed to $\gamma=0.4, 0.6, 0.8$. The differences in the distributions of daily returns of the stock indices shown in Figure 2 can thus be attributed to the

Figure 7: Probability density distribution of stable distributions for different stability parameters and location parameters. (i) Case with the scale parameter γ fixed at 1.0 and the stability parameters α = 1.2, 1.4, 1.6, (ii) Case with the stability parameter α fixed at 1.6 and the scale parameter γ = 0.4, 0.6, 0.8, In all cases, the skewness parameter β = 0.0, and the location parameter δ = 0.0.



scale parameter of the probability density function for the stable distribution.

5. Conclusions

In this study, we used the HMC method to perform a Bayesian estimation of the probability density function for the stable distribution of daily returns for the Nikkei, TOPIX, DJIA, and S&P500. No significant differences were found in the value of the stability parameter of the estimated probability density function for the stable distribution of the index data. Moreover, the distributions of the daily returns for the Nikkei and TOPIX were distorted, whereas those for the DJIA and S&P500 were not. It was also found that the difference in the distribution of the daily returns for the Japanese and US stock indices can be explained by the difference in the scale index of the probability density function for the stable distribution. Extending the method used in this study to high-frequency data would seem a reasonable next step. The analysis described here is based on daily data; different research results could be expected if high-frequency data are used. Moreover, even when the DE formula is used, the calculation time is quite long. Consequently, speeding up the calculation program is a priority. It would also be of interest to apply the HMC method to Bayesian estimation of probability density functions such as skewed normal distributions and skewed t distributions other than stable distributions and to identify the most suitable distribution function for the distribution of the daily returns for various stock indices.

Notes

1） For details, see Mandelbrot (1963), Fama (1965), Plerou *et al.* (1999), Gopikrishnan *et al.* (2000), Rachev and Mittnik (2000), and Gabaix *et al.* (2003).

2） For details, see Mantegna and Stanley (1995) and Gabaix *et al.* (2003).

3） For details, see Umeno (1998).

4） For details, see Lemke *et al.* (2015).

5） The HMC method was initially proposed by Duane *et al.* (1987) as an efficient way to generate gauge configurations in lattice quantum chromodynamics （QCD） calculations in particle physics. Later, it was applied to parameter estimation of stochastic processes in time series data. For details, see MacKay (2003, Chapter 30), Neal (1994), Neal (2011), Takaishi (2013), Nugroho and Morimoto (2015).

6） For details, see Nolan (2020).

7） For details, see Bremer *et al.* (2010).

8） For details, see Ooura (2005).

9） For details, see Gelman and Rubin (1992) and Gelman (1996).

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