Effective tools for analyzing the periodic motion of the generalized Liénard system

Makoto Hayashi*

1. Introduction

Many results concerning the qualitative properties of limit cycles of a classical Liénard equation:

$$\ddot{x} + f(x)\dot{x} + x = 0 \tag{1}$$

or the generalized Liénard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{2}$$

have been demonstrated in [LOP], [H2], [HVZ], [SV], [M1], [Zh], among others. These equations emerge in various scientific phenomena and have been extensively disseminated by mathematicians, physicists, economists and other researchers. The topics, as seen in the famous Hilbert 23rd problem, continue to be an area of active research.

Equation (1), known as the "Van der Pol equation" is derived from the oscillator circuit appears in technological phenomena. Equation (2) plays an important role in understanding the qualitative behavior of "natural phenomena". For instance, it is applied in various fields, such as the "Predator–Prey system" in biology, the "Hodgkin-Huxley system" in neurophysiology, the "Bogdanov-Takens system" which is essential to the development of bifurcation theory in mathematics, and the "Business cycle model in the Keynesian theory" in economic dynamics, among others.

In this study, we provide new tools for establishing criteria for the unique existence or the non-existence of periodic motions in the models described by the Liénard system (4), (6) or (9). Consequently, we assert that these results will contribute to understanding "economic phenomena" in the future.

T. Yasui [Ya] and T. Maruyama [Mt] have explained the important roles of equation (2) as derived from business cycle models. T. Owase [Ow] has historically introduced the mathematical methods for this equation. Recently, M. Galeotti–F. Gori [GG] and H. Murakami [M1]–[M4] have addressed the uniqueness and existence of limit cycles for equation (2) using the classical tools provided by Levinson–Smith [LS:1942] and Xiao–Zhang [XZ:2003]. Our

* College of Science and Technology, Nihon University

methods differ from theirs, offering a new direction, and we expect these approaches to influence future research.

In 1954, a tool for establishing the existence and uniqueness of the limit cycle for equation (1) was provided given by J. L. Massera [Ma]. This tool is based on the idea that the limit cycle is star-like (for the definition see [Ma] or [Zh]) and stable. It remains an important tool for equation (1) as demonstrated in the famous “Van der Pol equation” (see Example 1). However, Massera's method cannot be applied to the generalized equation (2) of (1). Although the result of [Ci] is a generalization of Massera’s theorem, it only covers the specific case of $g(x)=x$. Our aim in Section 2 is to provide a generalization of Proposition 1 for equation (2) and demonstrate its applicability several examples.

The improved Massera’method introduced in Section 2 is a significant tool and a powerfull method, as demonstrated in Examples 1–3. However, conditions (C3) or (C6) are quite stringent when used to determine the unique existence of limit cycles for equations (1) or (2). For instance, Theorem 2 cannot be applied to equations with $f(x)=3x^2-4x-3$ or $g(x)=x^{2n-1}(n\in\mathbb{N})$ in system (3). Our aim in Section 3 is to establish the uniqueness of the limit cycles for system (4) or (6) without replying on Massera’s conditions (C3) or (C6), which are equivalent to equation (2). Moreover, an example proposed by Duff-Levinson (introduced from a mathematical perspective) is discussed and the power of our tools is comfirmed. See [DL], [H1], or [H4].

In the final, we consider the non-existence of limit cycles for the generalized Liénard system (9). This aspect has not been studied in the context of business models. Establishing criteria for the non-existence of periodic motions in economic models is important. The effectiveness of these criteria is discussed with reference to the Bogdanov–Takens system ([H5], [MKKT], [P], [RW]) and the FitzHugh-Nagumo system ([Fi], [H6], [HH], [NAY], [Su]).

2. Tool 1 (Improved Massera’s method)†

Massera’s result [Ma] is as follows.

Proposition 1. *Assume that the conditions*

(C1) $f(x)$ *is a continuous function*,

(C2) *there exist* a *and* $b\,(a<0<b)$ *such that* $f(x)<0\ (a<x<b)$

and $f(x)>0\,(x\le a,\, x\ge b)$,

(C3) $f(x)$ is nondecreasing as $|x|$ increases

are satisfied. Then, equation (1) *has a unique limit cycle*.

As an important example of Proposition 1, we can show the following

Example 1. Consider equation (1) with $f(x)=\epsilon\,(x^2-1)$. This is the well-known as the “Van der Pol equation.” Note that condition (C3) is satisfied for this equation. Thus, the system has a unique limit cycle for all $\epsilon>0$.

To generalize the above proposition, we assume the following conditions

(C4) $f(x)$ and $g(x)$ are locally Lipschitz continuous functions, and $g(x)/x>0$,

(C5) $g(x)=xh(x)$, where $h(x)>0\ (x\neq 0)$.

Under the above conditions (C2) and (C4), the uniqueness of solutions of equation (2) for initial value problems

† This result is based on the paper: M. Hayashi, An improvement of Massera’s theorem for the Liénard equation, Applied Mathematical Sciences, **18** (2024), 179–184. https://doi.org/10.12988/ams.2024.918683.

is guaranteed, and the only equilibrium point $(0, 0)$ is unstable (for instance see [Le] or [Zh]).

Let $F(x) = \int_0^x f(\xi)d\xi$ and $G(x) = \int_0^x g(\xi)d\xi$. Our main result is the following

Theorem 2. *Assume that the conditions* (C2), (C4), (C5), *and additionally*

(C6) *$f(x)$ and $h(x)$ are nondecreasing as $|x|$ increases*,

(C7) $\limsup_{x\to\pm\infty} [G(x) \pm F(x)] = +\infty$

are satisfied. Then, equation (2) *has a unique limit cycle*, *which is stable and hyperbolic*.

Note that the system

$$\dot{x} = y, \qquad \dot{y} = -f(x)y - g(x) \tag{3}$$

or

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x) \tag{4}$$

is equivalent to equation (2). This is known as the Liénard system. Notably, the existence and non-existence of limit cycles in systems (3) or (4) coincide with those in equation (2).

Assume that there exist α and β such that $F(\alpha) = F(\beta) = F(0) = 0$ and $\alpha < 0 < \beta$, as given by condition (C2). Note that if such α and β do not exist, then system (4) has no limit cycles.

Let $G(\alpha) \geq G(\beta)$ without loss of generality, and let Ω be the domain surrounded by the closed plane curve $(1/2)y^2 + G(x) = G(\beta)$. From the fact (see [H1]) that the limit cycle of system (4) cannot exist within Ω due to conditions (C2) and (C4), we have the following

Corollary 3. *Under the conditions of Theorem 2*, *a limit cycle of system* (4) *must exist outside Ω and intersect both lines $x = \beta^*$ and $x = \beta$*, *where β^* is a negative number such that $G(\beta) = G(\beta^*)$*.

2.1 Proof

We prove Theorem 2 by the analogy to the discussion in [Ma]. The outline of the proof consists of three steps.

- Existence of a closed orbit Γ.

Under the conditions (C2), (C4), and (C7), it is known that system (4) has at least one closed orbit. For instance see [CV] or [Gr]. Thus, a closed orbit Γ of system (3) exists under these conditions.

- Γ is star-like.

By transforming system (4) to polar coordinates $x = \rho\cos\theta$ and $y = \rho\sin\theta$, the system is converted to the differential system with respect to ρ and θ:

$$\begin{cases} \dot{\rho} = \rho\cos\theta\sin\theta\{1 - h(\rho\cos\theta)\} - \rho\sin^2\theta f(\rho\cos\theta) \\ \dot{\theta} = -\cos^2\theta h(\rho\cos\theta) - \sin^2\theta - \sin\theta\cos\theta f(\rho\cos\theta). \end{cases}$$

Assume that Γ is not star-like. Then there exists a half-ray $\theta = \theta_0$ which intersects Γ at three points P_1, P_2, and P_3 on the xy-plane. Thus, $\dot{\theta}$ must change sign twice at these points as ρ increases. Given the monotonicity of $f(x)$ and $h(x)$, this is impossible. Therefore, Γ is star-like.

- Γ is unique.

In system (4) we consider a closed orbit Γ (which is star-like) and make the transformation $(x, y) \to (kx, ky)$

for $k \in \mathbb{R}^+$. Let $\Gamma\,(kx,\,ky) = \Gamma_k\,(x,\,y)$, and we obtain a family of simple closed curves Γ_k on the plane such that $\Gamma_k \supset \Gamma$ for $k > 1$, $\Gamma_k \subset \Gamma$ for $k < 1$ on both sides of Γ and $\Gamma_1 = \Gamma$.

Take a point $P(x,\,y)$ on Γ and consider its tangent vector $T(x,\,y)$. This is given by the slope, which in this case is

$$T(x,y) = -f(x) - \frac{g(x)}{y} = -f(x) - \frac{xh(x)}{y}.$$

From a similar transformation, the tangent vector $\widetilde{T}(x,y)$ at the point $P_k(kx,\,ky)$ on the closed curve Γ_k is parallel to the one at the point P. The slope of system (4) at this point is

$$T(kx,ky) = -f(kx) - \frac{g(kx)}{ky} = -f(kx) - \frac{xh(kx)}{y}.$$

From the monotonicity assumpitions on $f(x)$ and $h(x)$, the slope at the point P_k is given by

$$\widetilde{T}(P_k) = -f(kx) - \frac{xh(kx)}{y} \leq -f(x) - \frac{xh(x)}{y} = T(P_k) \;\text{ for } k > 1.$$

Similarly, we have

$$\widetilde{T}(P_k) = -f(kx) - \frac{xh(kx)}{y} \geq -f(x) - \frac{xh(x)}{y} = T(P_k) \;\text{ for } k < 1.$$

This implies that the orbit $\Gamma^+(P_k)$ of system (4) cannot move from inside (resp. outside) of Γ_k to outside (resp. inside) when $k > 1$ (resp. $k < 1$). Thus, system (4) cannot have any closed orbit other than Γ.

The mentioned facts above show that the only one limit cycle Γ is stable and hyperbolic. □

Remark. The case where conditions (C5) or (C6) is not satisfied has been discussed in Section 3.

2.2 Applications

We shall apply Theorem 2 to concrete systems.

Example 2. Consider system (4) with

$$\begin{cases} F(x) = \dfrac{1}{42}x^7 + \dfrac{m+n}{15}x^6 + \dfrac{m^2+4mn+n^2}{20}x^5 + \dfrac{m^2n+mn^2}{6}x^4 + \dfrac{m^2n^2}{6}x^3 + rx \\ g(x) = px^3 + qx, \end{cases}$$

where $m < 0 < n$, $r < 0$, $p > 0$ and $q > 0$.

This is equivalent to the Liénard equation $\ddot{x} + F'(x)\dot{x} + g(x) = 0$ in the form (2). Since $f'(x) = F''(x) = x\,(x+m)^2\,(x+n)^2$, we see that $f(x)$ and $h(x) = px^2 + q$ are nondecreasing as $|x|$ increases. Thus, we conclude from Theorem 2 that the system has a unique limit cycle. .

Obtaining the same result by other methods, for instance [H3] or [Zh] etc., is not easy. This method is straightforward and effective for this system.

Example 3. We consider the following system, known as a continuous piecewise linear Liénard system (PL):

$$F(x) = \begin{cases} T_R(x-u_R)+T_Cu_R \ (x \geq u_R) \\ T_Cx \ (u_L \leq x \leq u_R) \\ T_L(x-u_L)+T_Cu_L \ (x \leq u_L), \end{cases}$$

$$g(x) = \begin{cases} l_R(x-v_R)+l_Cv_R \ (x \geq v_R) \\ l_Cx \ (v_L \leq x \leq v_R) \\ l_L(x-v_L)+l_Cv_L \ (x \leq v_L), \end{cases}$$

where $T_R>0$, $T_C<0$, $T_L>0$, $l_R>0$, $l_C>0$, $l_L>0$, $u_L<0<u_R$ and $v_L<0<v_R$. Llibre-Ordóñez-Ponce [LOP] established the unique existence of the limit cycle for this system under the conditions $-u_L=u_R$ and $-v_L=v_R$. The solution orbits in this case have a special character. Our result improves upon this. Using Theorem 2, we have the following

Corollary 4. *System* (PL) *has a unique limit cycle if* $l_R \geq l_C>0$ and $l_L \geq l_C>0$.

3. Tool 2 (Method without Massera's condition) ‡

First, we consider the generalized equation of (2) with one parameter:

$$\ddot{x}+\lambda f(x)\dot{x}+g(x)=0, \tag{5}$$

where λ is a positive real number. For instance see [H1] and [H4].

Equation (5) is equivalent to the following Liénard system:

$$\dot{x}=y-\lambda F(x), \qquad \dot{y}=-g(x). \tag{6}$$

Throught this section, we assume the conditions (C4) and

(C8) there exist α and β with $\alpha<0<\beta$ such that $x(x-\alpha)(x-\beta)F(x)>0$.

Under the above assumptions that the uniqueness of solutions for system (6) in initial value problems is guaranteed, and the only equilibrium point $(0, 0)$ is unstable. For instance see [Le].

Let $G(x)=\displaystyle\int_0^x g(\xi)d\xi$ and assume $G(\alpha) \geq G(\beta)$ without loss of generality. Then there exists $\beta^* \in [\alpha, 0)$ such that $G(\beta)=G(\beta^*)$. Also, see Section 2.

The following result (see [Gr] or [CV]) is useful for the existence of the limit cycle.

Lemma 5. *If the conditions* (C4), (C7), *and* (C8) *are satisfied, then system* (6) *has at least one limit cycle.*

The following ([H1] or [HVZ]) is key to stating our result. Let Ω_1 be the region surrounded by the closed curve

‡ This result is based on the paper: M. Hayashi, A survey for a generalized Liénard equation without the Maseera's condition, International Journal of Applied Mathematics, **35** (2022), 611–623. https://dx.doi.org/10.12732/ijam.v35i4.9.

$V(x, y) = (\lambda/2)y^2 + G(x) = G(\beta)$. This curve surrounds the only equilibrium point $(0, 0)$ of system (6) under the conditions (C4) and (C8).

Lemma 6. *Assume the conditions* (C4), (C7), *and* (C8). *All limit cycles of system* (6) *exist outside* Ω_1 *and intersect the lines* $x = \beta^*$ *and* $x = \beta$.

Consider the case where there exists $r \neq 0$ such that $F''(r) = f'(r) = 0$. Note that the Massera's condition (C3) is not satisfied.

Let

$$p = \min\{p_i \in [\alpha, 0) \mid F'(p_i) = 0,\ F''(p_i) \neq 0, i \in \mathbb{N}\} < 0$$
$$\text{and } r = \min\{r_i < 0 \mid F''(r_i) = 0, i \in \mathbb{N}\}.$$

For the function $\Phi_\lambda(x) = \lambda F(x)^2 + 2G(x)$ and some $\lambda = \lambda_1 > 0$, we define

$$a_{\lambda_1} = \max\left\{ x \in [0, \beta] \,\middle|\, \Phi_{\lambda_1}(x) = \max_{\xi \in [0,\beta]} \left(\lambda_1 F(\xi)^2 + 2G(\xi)\right) \right\}.$$

Then, let

$$M_{\lambda_1} = \min\left\{ -\sqrt{\lambda_1 F(a_{\lambda_1})^2 + 2G(a_{\lambda_1})}\,, -\sqrt{2G(\beta)} \right\} < 0$$

and let $y_A(x)$ be the solution orbit starting from the point $A(0, M_{\lambda_1})$.

Lemma 7. *Under the conditions* (C4) *and* (C8), *if there exists a negative number* $x_1 = x_1(\lambda_1) \in (\alpha, 0]$ such that $y_A(x) < 0$ *for all* $x \in [x_1, 0]$, *then*

$$y_A(x_1) < M_{\lambda_1} + \int_0^{x_1} \frac{g(x)}{\lambda_1 F(x)} dx.$$

In fact, we have

$$y_A(x_1) = y_A(0) + \int_0^{x_1} \frac{-g(\xi)}{y_A(\xi) - \lambda_1 F(\xi)} d\xi < M_{\lambda_1} + \int_0^{x_1} \frac{g(\xi)}{\lambda_1 F(\xi)} d\xi.$$

Thus, we can choose x_1 such that the equation

$$M_{\lambda_1} + \int_0^{x_1} \frac{g(x)}{\lambda_1 F(x)} dx = 0 \tag{7}$$

is satisfied.

Lemma 8. *In the case where* $M_{\lambda_1} = -\sqrt{\lambda_1 F(a_{\lambda_1})^2 + 2G(a_{\lambda_1})}$, *the solution orbits* $(x(t), y(t))$ *cannot across the following curves* $C_1 \cup C_2$ *from the right side to the left side, where*

$$C_1 \;:\; y = \lambda_1 F(x) \quad \text{if } a_{\lambda_1} \le x \le \beta,$$
$$C_2 \;:\; V(x, y) = \frac{\lambda_1}{2} F(a_{\lambda_1})^2 + G(a_{\lambda_1}) \text{ and } y \le 0 \quad \text{if } 0 \le x \le a_{\lambda_1}.$$

In fact, we have $\dot{x}(t) = 0$ and $\dot{y}(t) < 0$ on C_1, and also

$$\dot{V}(x(t),y(t)) = -\lambda_1 g(x(t))F(x(t)) > 0$$

on C_2. See Figure 1 below.

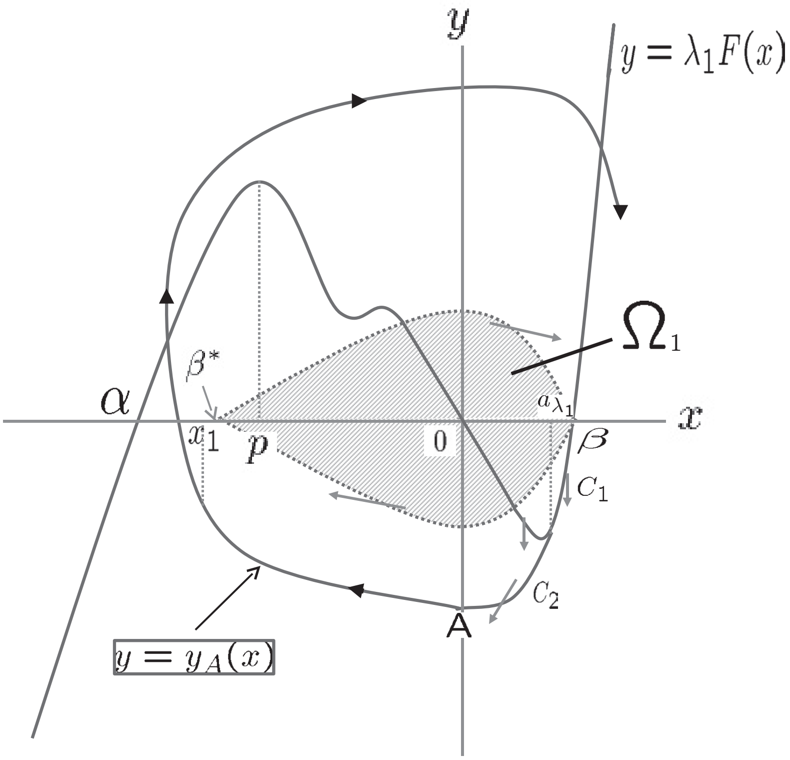

Figure 1 ($M_{\lambda_1} = -\sqrt{\lambda_1 F(a_{\lambda_1})^2 + 2G(a_{\lambda_1}}$)

In the case where $x_1 \leq p$, the following result is known from [H2] or [HT].

Theorem 9. *Let* $x_1 \leq p$. *Under the conditions* (C4), (C7), (C8), *and additionally*

(C9) $F(x)$ *is nondecreasing for* $x \leq x_1$ *and* $x \geq \beta$,

system (6) *has a unique limit cycle for all* $\lambda > 0$. *It intersects both the lines* $x = x_1$ *and* $x = \beta$, *and it is stable and hyperbolic*.

Note that the above result is independent of the existence of r.

If $\beta^* \leq p$, we do not need to calculate the above value x_1. In fact, the following result is given in [H3].

Proposition 10. *Let* $\beta^* \leq p$. *Under the conditions* (C4), (C7), (C8), *and additionally*

(C10) $F(x)$ is *nondecreasing for* $x \leq \beta^*$ *and* $x \geq \beta$,

system (6) *has a unique limit cycle for all* $\lambda > 0$. *It intersects both the lines* $x = \beta^*$ *and* $x = \beta$, *and it is stable and hyperbolic*.

Example 4. (the case of $\beta^* < p < r$)

Proposition 10 is applied to system (6) with $F(x) = x^3 + \dfrac{x^2}{2} - \dfrac{3}{2}x$ and $g(x) = x^{2n-1}$. In fact, this system has

$\beta^* = -\beta = -1 < p = -\dfrac{1+\sqrt{19}}{6} < r = -\dfrac{1}{6}$. According to Proposition 10, the above systems has a unique limit cycle without the need for calculating x_1.

Remark. In [H4], the results were given as $M_\lambda = -\sqrt{2G(\beta)}$. When λ is large, we have $M_\lambda = -\sqrt{\lambda F(a_\lambda)^2 + 2G(a_\lambda)}$. In this case, this method proves to be more useful than that in [H4].

We now consider the case of $p < x_1$ and provide the supplymental function

$$L(x; s) = \sqrt{\frac{1}{F(x)} \int_s^x \frac{g(\xi)}{F(\xi)} d\xi}$$

for some constant s. This function was defined in [H4].

Theorem 11. *Let* $p < x_1$. *Assume the conditions* (C4), (C7), (C8), *and additionally*
(C11) $F(x)$ *is nondecreasing for* $x \le p$ *and* $x \ge \beta$.

Then, system (6) *has a unique limit cycle for all* $\lambda \ge \lambda_1$ *if*

$$\lambda_1 \ge \lambda^* = \max_{x \in [p, x_1]} L(x; x_1).$$

It intersects both the lines $x = p$ *and* $x = \beta$, *and it is stable and hyperbolic*.

We shall prove the above theorem.

Let $y_A(x)$ be the solution orbit starting from the point $A(0, M_{\lambda_1})$. By solving the equation (7) for Lemma 7, there exists $x_1 \in (\alpha, 0]$ such that $y_A(x) \le 0$ for all $x \in [x_1, 0]$.

Let $y_B(x)$ be the solution orbit starting from the point $B(x_1, 0)$. If $x_1 \le p$, then $y_B(x)$ must intersect the curve $y = \lambda_1 F(x)$ for $x < x_1$. See Figure 1.

If $x_1 > p$ and $\lambda_1 \ge \lambda^*$, then, according to the Theorem in [H4], the orbit $y_B(x)$ must intersect the half segment $l = \{(x, y) \mid x = p, y \le \lambda_1 F(x)\}$. Thus, from Lemma 6, Lemma 8, and the uniqueness of solution orbits that the orbit $y_B(x)$ must encircle clockwise outside Ω_1 and cannot remain in the domain $\{(x, y) \mid p < x, y \in \mathbb{R}\}$. See Figure 2.

Therefore, we conclude, based on results from [H2] or [HT], that system (6) has at most one limit cycle, and this limit cycle is unique, as established in Lemma 6. Also see [H4] or [HVZ]. □

Note that the above theorem is independent of the existence of r ($p < r < \beta^*$ or $p < \beta^* \le r$). Namely, Massera's condition (C3) is not assumed.

Example 5. (the case of $p < r \le \beta^*$)

Consider system (6) with $F(x) = x^3 + 4x^2 - 5x$ and $g(x) = x^3$. Since $F'(x) = 3x^2 + 8x - 5$ and $F''(x) = 6x + 8$, the system has $\alpha = -5$ and $p = -\dfrac{4+\sqrt{31}}{3} < r = -\dfrac{4}{3} < \beta^* = -\beta = -1$.

We computed several values a_{λ_1}, M_{λ_1}, x_1, and λ^* using the computer algebra system Maple. Let $\lambda_1 = 0.3$. We have $a_{0.3} \fallingdotseq 0.5607$ and

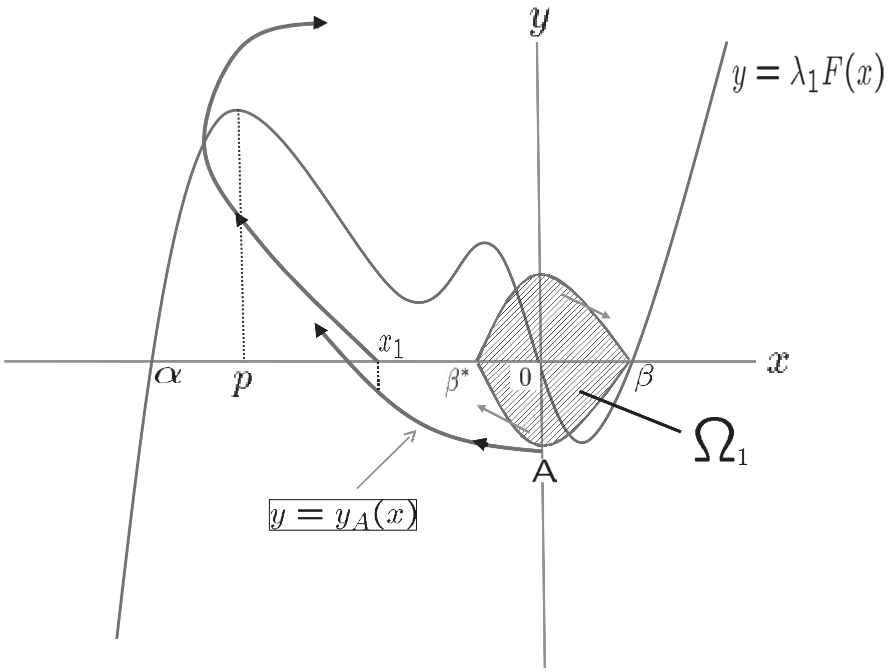

Figure 2 $\left(M_{\lambda_1} = -\sqrt{2G(\beta)} \right)$

$$M_{0.3} = -\sqrt{0.3F(a_{0.3})^2 + 2G(a_{0.3})} \fallingdotseq -\sqrt{0.61223} < -\sqrt{2G(\beta)} = -\frac{\sqrt{2}}{2}.$$

Solving the equation (8), we get $p < x_1 \fallingdotseq -1.8099 < \beta^*$ and

$$\lambda^* = \max_{x \in [p, x_1]} \sqrt{\frac{1}{F(x)} \int_0^x \frac{30\xi^2}{6\xi^4 + 45\xi^3 + 80\xi^2 - 90\xi - 270} d\xi} \fallingdotseq 0.2080 < \lambda_1 = 0.3.$$

Thus, we conclude from Theorem 11 that this system has a unique limit cycle for all $\lambda \geq 0.3$.

We consider the case where $G(\alpha) < G(\beta)$. Then there exists $\alpha^* \in (0, \beta]$ such that $G(\alpha) = G(\alpha^*)$. For some $\lambda = \lambda_2 > 0$, we set

$$q = \max\{q_j \in (0, \beta) \mid F'(q_j) = 0,\ F''(q_j) \neq 0, j \in \mathbb{N}\} > 0,$$
$$r^* = \max\{r_i > 0 \mid F''(r_i) = 0, i \in \mathbb{N}\},$$
$$b_{\lambda_2} = \min\left\{ x \in [\alpha, 0] \;\middle|\; \Phi_{\lambda_2}(x) = \max_{\xi \in [\alpha, 0]} \left(\lambda_2 F(\xi)^2 + 2G(\xi)\right) \right\}.$$

Then, let

$$M_{\lambda_2} = \max\{\sqrt{\lambda_2 F(b_{\lambda_2})^2 + 2G(b_{\lambda_2})}, \sqrt{2G(\alpha)}\} > 0$$

and $y_C(x)$ be the solution orbit starting from the point $C(0, M_{\lambda_2})$.

Lemma 12. *Under the conditions* (C4) *and* (C8), *if there exists a positive number* $x_2 = x_2(\lambda_2) \in [0, \beta)$ *such that* $y_C(x) > 0$ *for all* $x \in [0, x_2]$, *then*

$$y_C(x_2) > M_{\lambda_2} + \int_0^{x_2} \frac{g(x)}{\lambda_2 F(x)} dx.$$

We can choose x_2 such that the equation

$$M_{\lambda_2} + \int_0^{x_2} \frac{g(x)}{\lambda_2 F(x)} dx = 0 \tag{8}$$

is satisfied.

Corollary 13. *Let* $x_2 \geq q$. *Under the conditions* (C4), (C7), (C8), *and additionally*
(C12) $F(x)$ *is nondecreasing for* $x \leq \alpha$ *and* $x \geq x_2$,
system (6) *has a unique limit cycle for all* $\lambda > 0$. *It intersects both the lines* $x = \alpha$ *and* $x = x_2$, *and it is stable and hyperbolic*.

Corollary 14. *Let* $x_2 < q$. *Assume the conditions* (C4), (C7), (C8), *and additionally*
(C13) $F(x)$ *is nondecreasing for* $x \leq \alpha$ *and* $x \geq q$.
Then, system (6) *has a unique limit cycle for all* $\lambda \geq \lambda_2$ *if*

$$\lambda_2 \geq \lambda^{**} = \max_{x \in [x_2, q]} L(x; x_2).$$

It intersects both the lines $x = \alpha$ *and* $x = q$, *and it is stable and hyperbolic*.

3.1 An application

In 1964, Duff and Levinson([DL]) proposed in the following system:

$$\begin{cases} \dot{x} = y - \lambda \left(\dfrac{64}{35\pi} x^7 - \dfrac{112}{5\pi} x^5 + \dfrac{196}{3\pi} x^3 - \dfrac{C}{2} x^2 - \dfrac{36}{\pi} x \right) \\ \dot{y} = -x. \end{cases} \tag{DL}$$

System (DL) has three limit cycles if C is large enough and λ is small (see [Zh]). In 1997, [H1] provided a sufficient condition for λ such that system (DL) has exactly one limit cycle, as stated in the following

Proposition 15. *System* (DL) *with* $C = 45$ *has exactly one limit cycle for all* $\lambda \fallingdotseq 2.86896$.

Using the computer algebra system Maple, we find easily that system (DL) with $C = 45$ has $\alpha \fallingdotseq -0.38144$, $\beta \fallingdotseq 3.1798$, $r^* \fallingdotseq 2.2300$ and $q \fallingdotseq 2.694$, satisfying the conditions (C4), (C7), (C8), and (C10).

Let $\lambda (= \lambda_2) = 0.15$. We have $b_{0.15} \fallingdotseq -0.21735$ and

$$M_{0.15} = \sqrt{0.15 F(b_{0.15})^2 + 2G(b_{0.15})} \fallingdotseq \sqrt{0.2696} > \sqrt{2G(\alpha)} = -\alpha = 0.3814,$$

where $F(x) = \dfrac{64}{35\pi} x^7 - \dfrac{112}{5\pi} x^5 + \dfrac{196}{3\pi} x^3 - \dfrac{C}{2} x^2 - \dfrac{36}{\pi} x$ and $g(x) = x$.

Then, solving the equation (8)

$$\sqrt{0.2696}+\int_0^{x_2}\frac{x}{F(x)}dx=0,$$

we find $\alpha^* < x_2 \fallingdotseq 1.3995 < q$ and $\lambda^{**} \fallingdotseq 0.03478 < \lambda = 0.15$. Thus, we conclude from Corollary 14 the following

Theorem 16. *System* (DL) *with* $C = 45$ h*as exactly one limit cycle for all* $\lambda \geq 0.15$.

This is a dramatic improvement over Proposition 15.

4. Tool 3 (Method for Non-existence) §

We consider the Liénard–type system

$$\dot{x}=\frac{1}{a(x)}\left\{h(y)-\lambda F(x)\right\}, \qquad \dot{y}=-a(x)g(x), \tag{9}$$

where $a(x) > 0$ for all $x \in \mathbb{R}$. This system generalizes of the Liénard system (4) or (6) and was discussed in [AM] and [H4].

The aim in this section is to establish sufficient conditions for the non-existence of limit cycles in system (9) under the following

(C14) $a(x)$, $F(x)$, and $g(x)$ are locally Lipschitz continuous functions with $g(x)/x > 0$,

(C15) there exist $\alpha < 0$ such that $x(x-\alpha)F(x) > 0$ for $x > \alpha$,

(C16) $h(y)$ is a non-decreasing continuous function with $yh(y) > 0$.

Under the above assumptions, the uniqueness of solutions of system (9) for initial value problems is guaranteed, and the only equilibrium point $(0, 0)$ is stable.

In the case where $F'(0) > 0$, as indicated by (C15), the existence of a limit cycle is ruled out. Conversely, if $F'(0) < 0$, there exists a limit cycle for system (9) ensured by condition (C7).

Let $y = y(x)$ be the solution orbits starting from the point $P(x, h^{-1}F(x))$ on the characteristic curve $h(y) = F(x)$ and $x \geq \alpha^*$, where $G(x) = \int_0^x g(\xi)d\xi$ and α^* is a positive number such that $G(\alpha^*) = G(\alpha)$.

We introduce a supplemental function

$$P(x)=\int_0^x\frac{a^2(\xi)g(\xi)}{-h(-L(\alpha))+F(\xi)}d\xi-h^{-1}(F(x))-L(\alpha),$$

where

$$|H^{-1}(G(\alpha))|=L(\alpha)>0 \text{ and } H(y)=\int_0^y h(\xi)d\xi>0.$$

Our result is the following

Theorem 17. *If* $P(x) \leq 0$ *for all* $x \geq \alpha^*$, *then system* (9) *does not exhibit any limit cycles.*

§ This result is provided in the paper: M. Hayashi, A geometric criterion for the non-existence of the limit cycle of a Liénard-type system.

4.1 Outline of the proofs

The proof consists of the following three lemmas. Detailed explanations are provided in the submitted paper[§].

We consider the domain Ω^* surrounded by the closed curves $W(x, y) = H(y) + G(x) = G(\alpha)$, where $H(y) = \int_0^y h(\xi)d\xi$. It surrounds the only one equilibrium point O of the system.

Lemma 18. *Assume the conditions* (C14), (C15), *and* (C16). *Limit cycles of system* (9) *cannot reside within* Ω^*. *In other words, a solution orbit starting inside* Ω^* *cannot exist* Ω^* *through its boundary.*

The reason is similar to Lemma 6.

Lemma 19. *If* $y(0) \geq -L(\alpha)$, *then system* (9) *has no limit cycles.*

The proof derives from the geometric context provided in Lemma 18.

Lemma 20. *The inequality* $y(0) \geq -L(\alpha)$ *is equivalent to the inequality* $P(x) \leq 0$ *for all* $x \geq \alpha^*$.

Necessity is established by Lemma 19. Sufficiency is demonstrated as follows. Assuming that there exists $x_1 > \alpha^*$ such that $y_{x_1}(0) < -L(\alpha)$, we have

$$P(x_1) = \int_0^{x_1} \frac{a^2(\xi)g(\xi)}{-h(-L(\alpha)) + F(\xi)} d\xi - h^{-1}(F(x_1)) - L(\alpha) \leq 0$$

and there exists $x^* \in (0, x_1)$ such that $y_{x_1}(x^*) = -L(\alpha)$. Then, we have $P(x_1) > 0$. This contradicts $P(x_1) \leq 0$. □

Corollary 21. *If the inequalities* $P(\alpha^*) \leq 0$ *and*

$$P'(x) = \frac{a^2(x)g(x)}{-h(-L(\alpha)) + F(x)} - \frac{d}{dx} h^{-1}(F(x)) \leq 0$$

for all $x \geq \alpha^*$, *then system* (9) *has no limit cycles.*

4.2 Several examples

Theorem 17 can be applied to several systems of the form (9). By using Corollary 21, we can conclude that these systems do not exhibit limit cycles.

Example 6. Consider the system

$$\dot{x} = y - (x^2 + 2x), \qquad \dot{y} = -x^2 - 3x. \tag{10}$$

We have $F(x) = x^2 + 2x$ and $G(x) = \dfrac{x^3}{3} + \dfrac{3}{2}x^2$. It follows that there exists $\alpha^* > 1$ such that $G(-2) = G(\alpha^*)$. In this case, we find $P(1) < 0$ and $P'(x) < -\dfrac{1}{2}(3x + 1) < 0$ for all $x \geq 1$.

Example 7. Consider the system

$$\dot{x} = y^3 - (x^3 + 3x^2 + 2x), \qquad \dot{y} = -x. \tag{11}$$

We have $h(y) = y^3$, $F(x) = x^3 + 3x^2 + 2x$, and $G(x) = \dfrac{1}{2}x^2$. Given $h(y) = y^3$, $\alpha = -1$, $L(\alpha) = \sqrt[4]{2}$ and

$$P(x) = \int_0^x \frac{\xi}{\sqrt[4]{8} + \xi^3 + 3\xi^2 + 2\xi} d\xi - \sqrt[3]{x^3 + 3x^2 + 2x} - \sqrt[4]{2}.$$

It follows that $P(1) < 0$ and $P'(x) < 0$ for all $x \geq 1$.

Example 8. We consider the following system called Bogdanov–Takens system (for instance see [H5], [MKKT], [P], or [RW]) exhibiting a cusp of order 2:

$$\begin{cases} \dot{x} = y \\ \dot{y} = (x + \mu_2)y + x^2 + \mu_1, \end{cases} \tag{BT}$$

where μ_1 and μ_2 are real parameters. This system is a special case of the Liénard system (3) and is well-known in Bifurcation Theory. Applying Corollary 21, we have the following

Proposition 22. *System* (BT) *has no limit cycles if* $\mu_1 > -\mu_2^2$.

This result aligns with the partial result previously observed in bifurcation diagrams (for instance see [MKKT]).

Example 9. The two dimensional autonomous system is called the FitzHugh-Nagumo system ([HH], [Fi], [Na], [H1], [H6], and [Su] etc.) can be transformed into the Liénard system as follows.

$$\begin{cases} \dot{x} = y - \left\{ \dfrac{1}{3}x^3 + \eta x^2 + (\eta^2 - \eta_0^2)x \right\} \\ \dot{y} = -\dfrac{\rho b}{3} \left\{ x^3 + 3\eta x^2 + 3\left(\eta^2 + \dfrac{1}{b} - 1\right)x \right\} \end{cases} \tag{FHN}$$

By Corollary 21, we have the following

Proposition 23. *System* (FHN) *has no limit cycles under* $\eta^2 > \eta_0^2$ *and other conditions for* η^2 *and* η_0^2.

This result is partially consistent with the findings in [Su].

References

[AM] A.Aghajani and A.Moradifam (2009), “The generalized Liénard equations”, *Glasgow Mathematical Journal*, **51**, pp. 605–617.

[Ci] L. Ciambellotti (2009), “Uniqueness of limit cycles for Liénard systems. A generalization of Massera’s theorem”, *Qualitative Theory of Dynamical Systems*, **7**, pp. 405–410.

[CV] M.Cioni and G.Villari (2015), “An extension of Dragilev’s theorem for the existence of periodic solution of the Liénard equation”, *Nonlinear Analysis*, **20**, pp. 345–351.

[DL] G.F.D. Duff and N. Levinson (1952), “On the non-uniqueness of periodic solutions for an asymmetric Liénard equation”, *Quarterly Applied Mathematics*, **10**, pp. 86–88.

[Fi] R. FitzHugh (1961), “Impulses and physiological states in theoretical models of nerve membrane”, *Biophysical Journal*, **1**, pp. 445–466.

[GG] M. Galeotti and F. Gori (1989), “Uniqueness of periodic orbits in Liénard-type business-cycle models”, *Metoroeconomica*, **40**, pp. 135–146.

[Gr] J. Graef (1972), “On the generalized Liénard equation with negative damping”, *J. Differential Equations*, **12**, pp. 33–74.

[H1] M. Hayashi (1997), “On the uniqueness of the closed orbit of the Liénard system”, *Mathematica Japonica*, **47**, pp. 1–6.

［H2］ M. Hayashi (2016), "On the improved Massera's theorem for the unique existence of the limit cycle for a Liénard equation", *Rendi condi dell'Istituto di Matematica dell'Universita di Trieste*, **309**, pp. 211–226.

［H3］ M. Hayashi (2019), "A criterion for the unique existence of the limit cycle of a Liénard -type system with one parameter", *Advances in Dynamical Systems and Applications*, **14**, pp. 179–187.

［H4］ M. Hayashi (2021), "An improved criterion for the unique existence of the limit cycle of a Liénard -type system with one parameter", *Qualitative Theory of Dynamical Systems*, **20:25**.

［H5］ M. Hayashi (2004), "A global condition for the non-existence of limit cycles of Bogdanov–Takens system", *Far East Journal of Mathematical Sciences*, **14**, pp. 127–136.

［H6］ M. Hayashi (2000), "A note on the uniqueness of the closed orbit of FitzHugh–Nagumo system", *Quarterly Applied Mathematics*, **58**, pp. 171–176.

［HH］ A. L. Hodgkin and A. F. Huxley (1952), "A quantitative description of membrane current and its application to conduction and excitation in nerve", *Journal of Physiology*, **117**, pp. 500–544.

［HT］ M. Hayashi and M. Tsukada (2000), "A uniqueness theorem on the Liénard systems with a non-hyperbolic equilibrium point", *Dynamic Systems and Applications*, **9**, pp. 98–108.

［HVZ］ M. Hayashi, G. Villari and F. Zanolin (2018), "On the uniqueness of limit cycle for certain Liénard systems without symmetry", *Electronic Journal of Qualitative Theory of Differential Equations,* **55**, pp. 1–10.

［Le］ S. Lefschetz (1997), "Differential Equations: Geometric Theory", 2nd edn. *Dover Publications*, New York.

［LOP］ J. Llibre, M. Ordóñez and E. Ponce (2013), "On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems with symmetry", *Nonlinear Analysis*: *Real World Applications*, **14**, pp. 2002–2012.

［LS］ N. Levinson and O. K. Smith (1942), "A general equation for relaxation oscillations", *Duck Mathematical Journal*, **9**, pp. 382–403.

［Ma］ J. L. Massera (1954), "Sur un Théoreme de G. Sansone sur l'equation de Liénard", *Bollettino dell'Unione Matematica Italiana*, **9**, pp. 367–369.

［M1］ H. Murakami (2019), "A note on the "unique" business cycle in the Keynesian theory", *Metoroeconomica*, **70**, pp. 384–404.

［M2］ H. Murakami (2022), "The unique limit cycle in post Keynesian theory", *Chaos, Solitons and Fractals*, **154**, 111597.

［M3］ 村上 弘毅 (2021), "景気循環論における微分方程式の使用法", **経済学論（中央大学）, 61**, pp. 267–290.

［M4］ H. Murakami (2021), "On dynamics in a medium-term Keynesian model", *Discrete and continuous dynamical systems Series B*, **26**, pp. 5925–5940.

［MKKT］ T. Matsumoto, M. Komuro, H. Kokubu and R. Tokunaga (1993), "Bifurcations (Sights, Sounds and Mathematics)", *Springer–Verlag*, New York.

［Mt］ 丸山 徹 (2002), "景気変動の数理", **応用数理, 15**, pp. 77–80.

［NAY］ J. Nagumo, S. Arimoto and S. Yoshizawa (1962), "An active pulse transmission line simulating nerve axon", *Proceedings of IRE*, **50**, pp. 2061–2070.

［Ow］ 大和瀬 達二 (1989), "力学系理論と経済変動分析", **数理解析研究所講究録, 696**, pp. 222–233.

［P］ L. M. Perko (1992), "A global analysis of the Bogdanov–Takens system", *SIAM Journal on Applied Mathematics*, **52**, pp 1172–1192.

[RW] R. Roussarie and F. Wagener (1995), "A study of the Bogdanov–Takens bifurcation", *Resenhas*, **2**, pp. 1–25.

[Su] J. Sugie (1991), "Non-existence of periodic solutions for the FitzHugh nerve system", *Quarterly Applied Mathematics*, **49**, pp. 543–554.

[SV] M. Sabatini and G. Villari(2010), "On the uniqueness of limit cycles for Liénard equation: the legacy of G. Sansone", *LE MATEMATICHE*, **LXV**, pp. 201–214.

[Ya] 安井 琢磨 (1952), "自励振動と景気循環", **季刊理論経済学**, **4**, pp. 213–219.

[XZ] D. Xian and X. Zhang (2003), "On the uniqueness and non-existence of limit cycles for predator–prey systems", *Nonlinearity*, **16**, pp. 1185–1201.

[Zh] Z. Zhang, et. al. (1992), "Qualitative theory of differential equations", *Translations of Mathematical Monographs*, AMS, Providence, **102**.