Overview of Morrey spaces

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1 Introduction

The Sobolev space plays an important role in the theory of partial differential equations, combined with functional analysis. Smoothness of functions is measured by the norm of the Lebesgue space $L_p(\mathbb{R}^n)$ if the functions belong to the Sobolev space. The Triebel-Lizorkin space and Besov space are generalizations of the Sobolev space, and they can deal with smoothness of functions more precisely. In recent years it has turned out that the Morrey space is useful to measure the size of functions and compensate the deficit of the Lebesgue space. For example, the function $|x|^{-n/p}$ does not belong to the Lebesgue space $L_p(\mathbb{R}^n)$, but it belongs to the Morrey space $M_{p,q}(\mathbb{R}^n)$ if $0 < q < p < \infty$. Accordingly, we expect that investigation of functions by the Morrey space provides us with fruitful results. By replacing the Lebesgue space L_p with the Morrey space $M_{p,q}$ in the definitions of the Sobolev space, Triebel-Lizorkin space and Besov space, we obtain the definitions of the Sobolev-Morrey space, Triebel-Lizorkin-Morrey space and Besov-Morrey space, respectively.

Most results obtained for the L_p-based space can be carried over to the $M_{p,q}$-based space. For example, it is known that if $a(\xi)$ is a symbol of order 0, then the operator $a(D)$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. We can assert that $a(D)$ is also bounded on $M_{p,q}(\mathbb{R}^n)$ for $1 < q \leq p < \infty$. The key idea of extending the results for the Lebesgue space to the Morrey space is found in Peetre's paper [4]. First we split a function f into $f = f_0 + f_1$, where f_0 is supported in the ball of radius R centered at a point a, and f_1 vanishes in this ball. Then we apply the result for the Lebesgue space to f_0 and evaluate the term involved with f_1 by the Morrey norm.

The fundamental results related to the Morrey space are scattered in many papers, and some of them have only sketch of their proofs. The aim of this study is to collect the fundamental results on the Morrey space and give their proofs in detail so that they are easily accessible for those who want to apply Morrey spaces to their own studies. Some theorems will be proved in a way different from the known proofs. As analysis using the Morrey space is developing mathematical science, it will be useful to the study of economics.

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2 Basic properties of Morrey spaces

For $1 \le p \le \infty$ let $L_p(\Omega)$ be the space of p-integrable functions on a domain Ω of $\mathbb{R}^n$ with the norm $\|\cdot\|_{L_p(\Omega)}$. When $\Omega = \mathbb{R}^n$, we often write $\|f\|_{L_p}$ or $\|f\|_p$ for $\|f\|_{L_p(\mathbb{R}^n)}$.

Let $B(x, R)$ be the open ball with center $x \in \mathbb{R}^n$ and radius $R > 0$. For $0 < q \le p < \infty$ we set

$$\|f\|_{M_{p,q}} = \|f\|_{p,q} = \sup_B |B|^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(x)|^q \, dx \right)^{\frac{1}{q}}$$

for a measurable function f, where the supremum is taken over all balls B in $\mathbb{R}^n$. We define the Morrey space $M_{p,q}(\mathbb{R}^n)$ to be the space of all measurable functions f satisfying $\|f\|_{p,q} < \infty$. When $p = q$, the Morrey space $M_{p,q}$ coincides with the Lebesgue space $L_p(\mathbb{R}^n)$:

$$M_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n).$$

The Morrey space is sometimes denoted by $L_{q,\lambda}(\mathbb{R}^n)$ with $0 < q < \infty$ and $0 \le \lambda \le n$, which consists of all measurable functions f such that the norms

$$\|f\|_{q,\lambda} := \left(\sup_{R>0,\, x\in\mathbb{R}^n} R^{-\lambda} \int_{B(x,R)} |f(y)|^q \, dy \right)^{\frac{1}{q}}$$

are finite. It is easy to see that $M_{p,q}(\mathbb{R}^n) = L_{q,\lambda}(\mathbb{R}^n)$ if $\lambda = n(1 - q/p)$.

The scaling of f by $\lambda > 0$ is defined by

$$f_\lambda(x) = f(\lambda x).$$

For a ball $B = B(a, R)$ and $\lambda > 0$ we set $\lambda B = B(a, \lambda R)$. As observed from the following two lemmas, the index p plays a central role in the Morrey space $M_{p,q}$ rather than q.

Lemma 2.1. *For $\lambda > 0$ and $f \in M_{p,q}(\mathbb{R}^n)$ with $0 < q \le p < \infty$ we have*

$$\|f_\lambda\|_{M_{p,q}} = \lambda^{-\frac{n}{p}} \|f\|_{M_{p,q}}.$$

Proof. The change of variables $\lambda x = y$ gives

$$\begin{aligned} |B|^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(\lambda x)|^q \, dx \right)^{\frac{1}{q}} &= |B|^{\frac{1}{p}-\frac{1}{q}} \left(\lambda^{-n} \int_{\lambda B} |f(y)|^q \, dy \right)^{\frac{1}{q}} \\ &= \lambda^{-\frac{n}{p}} |\lambda B|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(B)}. \end{aligned}$$

Taking the supremum, we obtain the lemma. □

Lemma 2.2. *The function* $|x|^{-\frac{n}{p}}$ *belongs to* $M_{p,q}(\mathbb{R}^n)$ *for* $0 < q < p < \infty$, *although* $|x|^{-\frac{n}{p}} \notin L_p(\mathbb{R}^n)$.

Proof. Note that $|x|^{-n/p} \geq R^{-n/p}$ for $x \in B(0, R) =: B(R)$, and $|x|^{-n/p} \leq R^{-n/p}$ for $x \in B(0, R)^c$. In addition, $|B(a, R) \setminus B(0, R)| = |B(0, R) \setminus B(a, R)|$. Hence

$$\begin{aligned}\int_{B(a,R)} (|x|^{-\frac{n}{p}})^q\, dx &= \int_{B(a,R)\cap B(0,R)} |x|^{-\frac{nq}{p}}\, dx + \int_{B(a,R)\setminus B(0,R)} |x|^{-\frac{nq}{p}}\, dx \\ &\leq \int_{B(a,R)\cap B(0,R)} |x|^{-\frac{nq}{p}}\, dx + \int_{B(0,R)\setminus B(a,R)} |x|^{-\frac{nq}{p}}\, dx \\ &= \int_{B(0,R)} |x|^{-\frac{nq}{p}}\, dx = \frac{|B(1)|}{1-\frac{q}{p}} R^{n(1-\frac{q}{p})}.\end{aligned}$$

This implies that the Morrey norm of $|x|^{-\frac{n}{p}}$ is bounded by $|B(1)|^{1/p}(1-\frac{q}{p})^{-1/q}$. □

Lemma 2.3. *Let* $0 < q < q_1 \leq p < \infty$. *Then*

$$M_{p,q_1}(\mathbb{R}^n) \subset M_{p,q}(\mathbb{R}^n)$$

Proof. The lemma follows from Hölder's inequality or Jensen's inequality for convex functions:

$$\left(\frac{1}{|B|}\int_B |f(x)|^q\, dx\right)^{\frac{1}{q}} \leq \left(\frac{1}{|B|}\int_B |f(x)|^{q_1}\, dx\right)^{\frac{1}{q_1}}.$$

□

Lemma 2.4. *Let* $f \in M_{p,q}(\mathbb{R}^n)$ *with* $1 \leq q \leq p < \infty$. *Then for a ball B we have*

$$\int_B |f(y)|\, dy \leq |B|^{1-\frac{1}{p}} \|f\|_{M_{p,q}}.$$

Proof. Hölder's inequality gives

$$\int_B |f(y)|\, dy \leq |B|^{1-\frac{1}{q}} \|f\|_{L_q(B)} \leq |B|^{1-\frac{1}{p}} \|f\|_{p,q},$$

which is the desired result. □

Lemma 2.5. *For* $1 \leq q \leq p < \infty$ *we have* $M_{p,q}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ *by defining the pairing*

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx \qquad \text{for } f \in M_{p,q}(\mathbb{R}^n),\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Remark. It is obvious that $f \in M_{p,q}$ is a Schwartz distribution for $1 \leq q$, since $M_{p,q} \subset L_1^{loc}$. The lemma asserts more strongly that f is a tempered distribution.

Proof. Let $\Omega_0 = B(0, 1)$ and $\Omega_j = B(0, 2^j) \setminus B(0, 2^{j-1})$ for $j \in \mathbb{N}$. Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $|\varphi(x)| \lesssim (1+|x|)^{-n}$, we have

$$\int_{\mathbb{R}^n} |f(x)\varphi(x)|\,dx \lesssim \sum_{j=0}^{\infty} \int_{\Omega_j} \frac{|f(x)|}{(1+|x|)^n}\,dx \le \sum_{j=0}^{\infty} 2^{-n(j-1)} \int_{|x|\le 2^j} |f(x)|\,dx$$
$$\le \sum_{j=0}^{\infty} 2^{-n(j-1)} |B(0,2^j)|^{1-\frac{1}{p}} \|f\|_{p,q} \lesssim \sum_{j=0}^{\infty} 2^{-\frac{jn}{p}} \|f\|_{p,q}.$$

This implies that f is a bounded linear functional on $\mathcal{S}$. □

Lemma 2.6. *Let* $0<q<p<\infty$. *The space* $M_{p,q}(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$ *is not dense in the Morrey space* $M_{p,q}(\mathbb{R}^n)$. *Accordingly, neither* $C_0^\infty(\mathbb{R}^n)$ *nor* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $M_{p,q}(\mathbb{R}^n)$.

Proof. This lemma is found in [7, p25], but the proof is omitted. Set $f(x)=|x|^{-n/p}$, and let $g\in M_{p,q}(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$. Choose ϵ so that $\epsilon^{-n/p}=\|g\|_{L_\infty}$. Then we have, for $|x|\le 2^{-p/n}\epsilon$,

$$|f(x)-g(x)| \ge |x|^{-n/p}-\epsilon^{-n/p} \ge \tfrac{1}{2}|x|^{-n/p}.$$

Hence, with $B(r)=B(0,r)$ for $r>0$,

$$\|f-g\|_{M_{p,q}} \ge |B(2^{-\frac{p}{n}}\epsilon)|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{B(2^{-\frac{p}{n}}\epsilon)} |x|^{-nq/p}\,dx\right)^{1/q} = \frac{|B(1)|^{\frac{1}{p}}}{(1-\frac{q}{p})^{\frac{1}{q}}}.$$

Since the last constant depends only on n, p and q, we obtain the lemma. □

Lemma 2.7. *Let* $1\le q\le p<\infty$. *A polynomial* $P(x)$ *belongs to* $M_{p,q}(\mathbb{R}^n)$ *if and only if* $P(x)=0$.

Proof. Assume that $P(x)$ is a polynomial and not identically 0. Then there exist $R_0>0$, $\delta>0$ and $\omega\in\mathbb{R}^n$ with $|\omega|=1$ such that $|P(x)|\ge 1$ if $|x|\ge R_0$ and $|x/|x|-\omega|\le\delta$. Let E be the cone defined by

$$E=\{x\in\mathbb{R}^n\setminus\{0\} : |x/|x|-\omega|\le\delta\}.$$

For $R\ge 2R_0$ we have

$$\|P\|_{L_q(B(0,R))} \ge |(B(0,R)\setminus B(0,R/2))\cap E|^{\frac{1}{q}},$$

which gives

$$\|P\|_{p,q} \ge |B(0,R)|^{\frac{1}{p}-\frac{1}{q}} \|P\|_{L_q(B(0,R))} \gtrsim |B(0,R)|^{\frac{1}{p}}.$$

Since $|B(0,R)|\to\infty$ as $R\to\infty$, we have $P\notin M_{p,q}(\mathbb{R}^n)$. □

3 Hardy-Littlewood maximal operators

For a locally integrable function f we set

$$Mf(x) = \sup_{R>0} \frac{1}{|B(0,R)|} \int_{B(x,R)} |f(y)|\, dy.$$

We call M the Hardy-Littlewood maximal operator.

Theorem 3.1. *(i) Let* $1 < p < \infty$. *M is bounded on* $L_p(\mathbb{R}^n)$ *and satisfies*

$$\|Mf\|_{L_p} \le 2\left(\frac{3^n p}{p-1}\right)^{\frac{1}{p}} \|f\|_{L_p}.$$

(ii) For $f \in L_1(\mathbb{R}^n)$ *and* $\lambda > 0$ *we have*

$$|\{Mf > \lambda\}| \le \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Proof. See Theorem 1.45 and Corollary 1.3 in [6]. We note that (i) follows from (ii) and the obvious inequality $\|Mf\|_\infty \le \|f\|_\infty$ by using the Marcinkiewicz interpolation theorem.

Theorem 3.2 (Chiarenza-Frasca 1987)**.** *(i) Let* $1 < q \le p < \infty$. *Then M is bounded on* $M_{p,q}(\mathbb{R}^n)$ *and satisfies*

$$\|Mf\|_{M_{p,q}} \lesssim_{n,p,q} \|f\|_{M_{p,q}}.$$

(ii) Let $1 \le p < \infty$. *Then for* $\lambda > 0$ *and a ball B*

$$\big|\{Mf > \lambda\} \cap B\big| \lesssim_{n,p} \frac{|B|^{1-\frac{1}{p}}}{\lambda} \|f\|_{M_{p,1}}.$$

Proof. The proof by Chiarenza-Frasca [1] uses the dual inequality of Stein-type (see [6, Corollary 1.3, p112]). For (i) we give here another proof which uses the L_p boundedness of M. Take $x_0 \in \mathbb{R}^n$ and $R > 0$ arbitrarily, and set

$$\Omega_0 = B(x_0, 2R), \qquad \Omega_j = B(x_0, 2^{j+1}R) \setminus B(x_0, 2^j R) \quad \text{for } j \in \mathbb{N}. \tag{3.1}$$

In what follows we write $B(x_0, R)$ simply as $B(R)$. Following the method of Tang-Xu [11], we set $f_j = f\chi_{\Omega_j}$ and decompose f as

$$f = \sum_{j=0}^{\infty} f_j.$$

Since $Mf \le \sum_j Mf_j$, Minkowski's inequality gives

$$\left(\int_{B(R)} (Mf)^q\,dx\right)^{\frac{1}{q}} \le \sum_{j=0}^{\infty}\left(\int_{B(R)} (Mf_j)^q\,dx\right)^{\frac{1}{q}}.$$

For f_0 we use the L_p boundedness of M to get

$$\|Mf_0\|_{L_q(B(R))} \lesssim \|f\|_{L_q(B(2R))} \le |B(2R)|^{-\frac{1}{p}+\frac{1}{q}}\|f\|_{p,q}.$$

Let $j \ge 1$. For $x \in B(R)$ we have

$$(Mf_j)(x) \le \frac{1}{|B(2^jR)|}\int_{\Omega_j}|f(y)|\,dy \le \frac{|B(2^{j+1}R)|^{1-\frac{1}{p}}}{|B(2^jR)|}\|f\|_{p,q} \lesssim |B(2^jR)|^{-\frac{1}{p}}\|f\|_{p,q}.$$

Hence

$$\|Mf_j\|_{L_q(B(R))} \lesssim |B(R)|^{\frac{1}{q}}|B(2^jR)|^{-\frac{1}{p}}\|f\|_{p,q} \lesssim 2^{-\frac{jn}{p}}|B(R)|^{-\frac{1}{p}+\frac{1}{q}}\|f\|_{p,q}.$$

Combining the above inequalities, we get

$$\|Mf\|_{p,q} \lesssim \left(1+\sum_{j=1}^{\infty}2^{-\frac{jn}{p}}\right)\|f\|_{p,q},$$

which yields the theorem.

For (ii) we follow the method of Chiarenza-Frasca [1]. For a weight w, i.e. a non-negative function on $\mathbb{R}^n$, we denote by $w(D)$ for a measurable subset D. We use the same notation as in the proof of (i). Applying the dual inequality of Stein-type (see [6, Theorem 1.45]) to the weight $w = \chi_{B(R)}$, we have

$$w(Mf > \lambda) = w(\{x:\ Mf(x) > \lambda\}) \le \frac{3^n}{\lambda}\int_{\mathbb{R}^n}|f(x)|M\chi_{B(R)}(x)\,dx.$$

By definition of the maximal function we find that

$$\begin{aligned} &M\chi_{B(R)}(x) \lesssim 1 \quad \text{for } |x-x_0| \le 2R;\\ &M\chi_{B(R)}(x) \lesssim_n \frac{R^n}{(|x-x_0|-R)^n} \quad \text{for } |x-x_0| \ge 2R. \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^n}|f(x)|M\chi_{B(R)}(x)\,dx &\lesssim \int_{B(2R)}|f(x)|\,dx + \sum_{j=1}^{\infty}\int_{B(2^{j+1}R)\setminus B(2^jR)}\frac{|f(x)|R^n}{(|x-x_0|-R)^n}\,dx\\ &\lesssim |B(2R)|^{1-\frac{1}{p}}\|f\|_{p,1} + \sum_{j=1}^{\infty}\int_{B(2^{j+1}R)}\frac{|f(x)|R^n}{(2^{j-1}R)^n}\,dx\\ &\lesssim |B(R)|^{1-\frac{1}{p}}\|f\|_{p,1} + \sum_{j=1}^{\infty}2^{-jn}|B(2^{j+1}R)|^{1-\frac{1}{p}}\|f\|_{p,1}\\ &\lesssim |B(R)|^{1-\frac{1}{p}}\|f\|_{p,1}\left(1+\sum_{j=1}^{\infty}2^{-\frac{jn}{p}}\right). \end{aligned}$$

On the other hand,

$$w(Mf>\lambda)=\int_{Mf>\lambda}\chi_{B(R)}(x)\,dx=\big|\{Mf>\lambda\}\cap B(R)\big|.$$

Summing up, we obtain (ii). □

Theorem 3.3 (Fefferman-Stein 1971)**.** *Let $1<p<\infty$ and $1<r\le\infty$. Let $\{f_j\}$ be a sequence of measurable functions on $\mathbb{R}^n$. Then*

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^r\right)^{\frac{1}{r}}\right\|_{L_p}\lesssim_{n,p,r}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L_p}.$$

Proof. This theorem is the vector-valued maximal inequality, which was first obtained by Fefferman-Stein [2]. See [6, Theorem 1.49] for a simpler proof. □

Theorem 3.4 (Tang-Xu 2005)**.** *Let $1<q\le p<\infty$ and $1<r\le\infty$. Let $\{f_j\}$ be a sequence of measurable functions on $\mathbb{R}^n$. Then*

$$\left\|\left(\sum_{j=1}^{\infty}(Mf_j)^r\right)^{\frac{1}{r}}\right\|_{M_{p,q}}\lesssim_{n,p,q,r}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M_{p,q}}.$$

Proof. We follow the method of Tang-Xu [11]. Let $x_0\in\mathbb{R}^n$ and $R>0$ fixed arbitrarily, and define Ω_i with $i\in\mathbb{N}_0$ as in (3.1). We decompose f_j as

$$f_j=\sum_{i=0}^{\infty}f_j^i,\qquad f_j^i=f_j\chi_{\Omega_i}.$$

For $i=0$ we have, by Theorem 3.3,

$$\big\|\|\{Mf_j^0\}\|_{l_r}\big\|_{L_q(B(R))}\lesssim_{q,r}\big\|\|\{f_j\}\|_{l_r}\big\|_{L_q(B(2R))}\le|B(2R)|^{-\frac{1}{p}+\frac{1}{q}}\big\|\|\{f_j\}\|_{l_r}\big\|_{p,q}.$$

For $i\ge1$ we use Minkowski's inequality twice to get

$$\begin{aligned}&\left\{\int_{B(R)}\left(\sum_{j=1}^{\infty}\left[M\left(\sum_{i=1}^{\infty}f_j^i\right)\right]^r\right)^{\frac{q}{r}}dx\right\}^{\frac{1}{q}}\le\left\{\int_{B(R)}\left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}Mf_j^i\right)^r\right)^{\frac{q}{r}}dx\right\}^{\frac{1}{q}}\\&\le\left\{\int_{B(R)}\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}(Mf_j^i)^r\right)^{\frac{1}{r}}\right)^{q}dx\right\}^{\frac{1}{q}}\le\sum_{i=1}^{\infty}\left\{\int_{B(R)}\left(\sum_{j=1}^{\infty}(Mf_j^i)^r\right)^{\frac{q}{r}}dx\right\}^{\frac{1}{q}}.\end{aligned}$$

Here we used $\|\{\|\{a_{ij}\}_i\|_{l_1}\}_j\|_{l_r} \le \|\{\|\{a_{ij}\}_j\|_{l_r}\}_i\|_{l_1}$ for a double-indexed sequence $\{a_{ij}\}_{i,j}$ in the second inequality, and $\|\sum_i g_i\|_q \le \sum_i \|g_i\|_q$ in the third inequality.

For $i \ge 1$ and $x \in B(R)$ we have

$$\begin{aligned}
\left(\sum_{j=1}^{\infty}(Mf_j^i)^r(x)\right)^{\frac{1}{r}} &\lesssim \left(\sum_{j=1}^{\infty}\left\{\frac{1}{|B(2^{i-1}R)|}\int_{\Omega_i}|f_j(y)|\,dy\right\}^r\right)^{\frac{1}{r}} \\
&\le \frac{1}{|B(2^{i-1}R)|}\int_{B(2^{i+1}R)}\left(\sum_{j=1}^{\infty}|f_j(y)|^r\right)^{\frac{1}{r}}dy \\
&\le \frac{|B(2^{i+1}R)|^{1-\frac{1}{q}}}{|B(2^{i-1}R)|}\left\|\|\{f_j\}\|_{l_r}\right\|_{L_q(B(2^{i+1}R))} \\
&\lesssim 2^{-\frac{in}{p}}|B(R)|^{-\frac{1}{p}}\left\|\|\{f_j\}\|_{l_r}\right\|_{p,q}.
\end{aligned}$$

Therefore

$$\left\|\|\{M\left(\sum_{i=1}^{\infty}f_j^i\right)\}_j\|_{l_r}\right\|_{L_q(B(R))} \lesssim \sum_{i=1}^{\infty}2^{-\frac{in}{p}}|B(R)|^{-\frac{1}{p}+\frac{1}{q}}\left\|\|\{f_j\}\|_{l_r}\right\|_{p,q}.$$

Combining the estimates for $i = 0$ and $i \ge 1$, we obtain the theorem.

4 Littlewood-Paley theory

We choose a C^∞ function ψ whose Fourier transform $\mathcal{F}\psi(\xi) = \hat{\psi}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\xi}\psi(x)\,dx$ satisfies

$$\chi_{B(0,1)} \le \hat{\psi} \le \chi_{B(0,2)}, \tag{4.1}$$

where χ_D denotes the characteristic function of a subset $D \subset \mathbb{R}^n$. We define a family of C^∞ functions $\{\phi_j\}_{j\in\mathbb{Z}}$ by

$$\hat{\phi}_j(\xi) = \hat{\psi}(2^{-j}\xi) - \hat{\psi}(2^{1-j}\xi). \tag{4.2}$$

We note that these functions satisfy the following properties:

$$\hat{\phi}_j \ge 0, \qquad \operatorname{supp}\hat{\phi}_j \subset \{\xi \,|\, 2^{j-1} \le |\xi| \le 2^{j+1}\}, \qquad \hat{\phi}_j(\xi) = \hat{\phi}_0(2^{-j}\xi),$$

$$\hat{\psi}(\xi) + \sum_{j=1}^{\infty}\hat{\phi}_j(\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^n, \qquad \sum_{j=-\infty}^{\infty}\hat{\phi}_j(\xi) = 1 \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}.$$

It is convenient to define the spaces and norms of vector-valued functions

$$\{f_j\}_{j\in\mathbb{Z}} = (\cdots, f_{-2}(x), f_{-1}(x), f_0(x), f_1(x), f_2(x), \cdots)$$

on $\mathbb{R}^n$. Let $0 < q \le p < \infty$ and $0 < r \le \infty$. We denote by $M_{p,q}(\mathbb{R}^n, l_r)$ the space of all l_r-valued functions $\{f_j\}_{j\in\mathbb{Z}}$ satisfying $\|\{f_j\}\|_{l_r} := (\sum_{j\in\mathbb{Z}}|f_j|^r)^{\frac{1}{r}} \in M_{p,q}(\mathbb{R}^n)$, and set

$$\|\{f\}_j\|_{M_{p,q}(l_r)} = \|\|\{f_j\}\|_{l_r}\|_{M_{p,q}} \tag{4.3}$$

with the usual modification for $r=\infty$. The space $L_p(\mathbb{R}^n, l_r)$ and the norm $\|\{f_j\}\|_{L_p(l_r)}$ are defined similarly.

The Littlewood-Paley theory for L_p spaces, which is stated below, asserts that $L_p(\mathbb{R}^n)$ is isomorphic to $L_p(\mathbb{R}^n, l_2)$ in some sense.

Theorem 4.1. *Let* $1<p<\infty$. *Then*

$$\left\| \|\{\phi_j * f\}_{j\in\mathbb{Z}}\|_{l_2} \right\|_{L_p} \sim \|f\|_{L_p}.$$

In other words,

$$\left[\int_{\mathbb{R}^n} \left(\sum_{j=-\infty}^{\infty} |\phi_j * f|^2 \right)^{\frac{p}{2}} dx \right]^{\frac{1}{p}} \sim \|f\|_{L_p}.$$

Proof. See [6, Theorem 3.2] □

Theorem 4.2 (Mazzucato 2003). *Let* $1<q\le p<\infty$. *Then*

$$\left\| \left(\sum_{j=-\infty}^{\infty} |\phi_j * f|^2 \right)^{\frac{1}{2}} \right\|_{M_{p,q}} \sim \|f\|_{M_{p,q}}.$$

In other words,

$$\sup_B |B|^{-\frac{1}{p}+\frac{1}{q}} \left[\int_B \left(\sum_{j=-\infty}^{\infty} |\phi_j * f|^2 \right)^{\frac{q}{2}} dx \right]^{\frac{1}{q}} \sim \|f\|_{M_{p,q}}.$$

Theorem 4.2, which extends Theorem 4.1 to Morrey spaces, can be proved in the same line of the proof of Theorem 4.1, although we need to avoid use of the duality argument that does not work for Morrey spaces. For the proof of Theorem 4.2 we prepare some lemmas which are involved with the maps between $M_{p,q}(\mathbb{R}^n, l_2)$ and $M_{p,q}(\mathbb{R}^n)$.

Lemma 4.3 (Khinchine's inequality). *Let* $\{r_j(t)\}_{j\in\mathbb{N}}$ *be the Rademacher sequence of functions defined on* $[0, 1]$. *Let* $0<p<\infty$. *For* $\{a_j\}_{j\in\mathbb{N}} \in l_2$ *we have*

$$\left\| \sum_{j=1}^{\infty} a_j r_j \right\|_{L_p(0,1)} \sim \|\{a_j\}\|_{l_2}.$$

Proof. See [6, Theorem 3.1]. □

Lemma 4.4. *Let $\{r_j(t)\}_{j\in\mathbb{Z}}$ be the Rademacher sequence of functions, which is rearranged so that the subscripts j range over $\mathbb{Z}$. Then*

$$\left|\sum_{j\in\mathbb{Z}} r_j(t)\phi_j(x)\right| \le \sum_{j\in\mathbb{Z}} |\phi_j(x)| \lesssim_n |x|^{-n} \tag{4.4}$$

for $t\in[0,1]$ and $x\neq 0$.

Proof. The first inequality is obvious by $|r_j(t)|\le 1$. Since $|\phi_0(x)|\le C(1+|x|)^{-n-1}$ and $\phi_j(x)=2^{jn}\phi_0(2^jx)$, we have

$$J := |x|^n \sum_j |\phi_j(x)| \lesssim \sum_j \frac{(2^j|x|)^n}{(1+|2^jx|)^{n+1}}.$$

Choose $l\in\mathbb{Z}$ so that $2^{-l-1}<|x|\le 2^{-l}$. Then

$$J \lesssim \sum_{j=-\infty}^{l} 2^{n(j-l)} + \sum_{j=l+1}^{\infty} \frac{2^{n(j-l)}}{2^{(n+1)(j-l-1)}} \le 2^{n+1}\sum_{j\in\mathbb{Z}} 2^{-|j-l|} \lesssim \sum_{j\in\mathbb{Z}} 2^{-|j|},$$

which yields the lemma. □

Lemma 4.5. *Let $1<q\le p<\infty$. If $f\in M_{p,q}(\mathbb{R}^n)$, then*

$$\left\|\left(\sum_{j\in\mathbb{Z}} |\phi_j * f|^2\right)^{\frac{1}{2}}\right\|_{M_{p,q}} \lesssim_{n,p,q} \|f\|_{M_{p,q}}.$$

Proof. Let $x_0\in\mathbb{R}^n$ and $R>0$ fixed arbitrarily. Decompose f as

$$f=f_0+f_1, \qquad f_0=f\chi_{B(2R)}, \quad f_1=f(1-\chi_{B(2R)}),$$

where $B(2R)=B(x_0,2R)$. The result for L_p spaces gives

$$\int_{B(R)}\Bigl(\sum_j |\phi_j * f_0|^2\Bigr)^{\frac{q}{2}}dx \lesssim \int_{B(2R)} |f|^q\,dx \le \Bigl(|B(2R)|^{-\frac{1}{p}+\frac{1}{q}}\|f\|_{p,q}\Bigr)^q.$$

For f_1 we use Theorem 4.3 with a suitably arranged Rademacher's sequence to get

$$\int_{B(R)}\left(\sum_j |\phi_j * f_1(x)|^2\right)^{\frac{q}{2}}dx \lesssim \int_0^1 dt \int_{B(R)}\left|\sum_j r_j(t)\phi_j * f_1(x)\right|^q dx.$$

We note that

$$|x-y|\ge |y-x_0|-|x-x_0|\ge 2^{i-1}R,$$

if $x\in B(R)=B(x_0,R)$ and $|y-x_0|\ge 2^iR$ with $i\in\mathbb{N}$. For $x\in B(R)$ we have, by Lemma 4.4,

$$\begin{aligned}\left|\sum_j r_j(t)\phi_j * f_1(x)\right| &\lesssim \int_{|y-x_0|>2R} |x-y|^{-n}|f(y)|\,dy \\ &\lesssim \sum_{i=1}^{\infty} \int_{2^iR<|y-x_0|\le 2^{i+1}R} (2^iR)^{-n}|f(y)|\,dy \\ &\lesssim \sum_{i=1}^{\infty} (2^iR)^{-n}|B(2^{i+1}R)|^{1-\frac{1}{p}}\|f\|_{p,q} \\ &\lesssim |B(R)|^{-\frac{1}{p}}\|f\|_{p,q}\sum_{i=1}^{\infty} 2^{-\frac{in}{p}}.\end{aligned}$$

Hence

$$\int_{B(R)}\left(\sum_j |\phi_j * f_1(x)|^2\right)^{\frac{q}{2}} dx \lesssim |B(R)|\left(|B(R)|^{-\frac{1}{p}}\|f\|_{p,q}\right)^q = \left(|B(R)|^{-\frac{1}{p}+\frac{1}{q}}\|f\|_{p,q}\right)^q.$$

Combining the estimates for f_0 and f_1, we obtain the lemma. □

Lemma 4.6. *If $\{f_j\}_{j\in\mathbb{Z}} \in M_{p,q}(\mathbb{R}^n, l_2)$, then the series $\sum_{j\in\mathbb{Z}}\phi_j * f_j =: T(\{f_j\})$ converges in $M_{p,q}(\mathbb{R}^n)$, and the map $\{f_j\}_{j\in\mathbb{Z}} \mapsto T(\{f_j\})$ is a bounded linear operator from $M_{p,q}(\mathbb{R}^n, l_2)$ to $M_{p,q}(\mathbb{R}^n)$, i.e.*

$$\Big\|\sum_{j\in\mathbb{Z}}\phi_j * f_j\Big\|_{M_{p,q}} \lesssim_{n,p,q} \|\|\{f_j\}\|_{l_2}\|_{M_{p,q}}.$$

Proof. The assertion on the $M_{p,q}$ convergence can be seen from the fact that the calculations below hold for the sum in j ranging over any subset of $\mathbb{Z}$.

Let $x_0\in\mathbb{R}^n$ and $R>0$ fixed arbitrarily. We decompose f_j as

$$f_j = f_j^0 + f_j^1, \qquad f_j^0 = f_j\chi_{B(2R)}, \qquad f_j^1 = f_j(1-\chi_{B(2R)}),$$

where $B(2R) = B(x_0, 2R)$. From the result for L_p spaces we have

$$\|T(\{f_j^0\})\|_{L_q(B(R))} \lesssim \|\{f_j^0\}\|_{L_q(\mathbb{R}^n,l_2)} \le \|\{f_j\}\|_{L_q(B(2R),l_2)} \lesssim |B(R)|^{-\frac{1}{p}+\frac{1}{q}}\|\{f_j\}\|_{M_{p,q}(l_2)},$$

which gives

$$\|T(\{f_j^0\})\|_{M_{p,q}} \lesssim \|\{f_j\}\|_{M_{p,q}(l_2)}.$$

For $x\in B(R) = B(x_0, R)$ we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\phi_j * f_j^1(x)| &\le \sum_{j\in\mathbb{Z}}\int_{|y-x_0|>2R}|\phi_j(x-y)f_j^1(y)|\,dy \\ &\le \int_{|y-x_0|>2R}\left(\sum_j|\phi_j(x-y)|^2\right)^{\frac{1}{2}}\left(\sum_j|f_j^1(y)|^2\right)^{\frac{1}{2}} dy.\end{aligned}$$

Since the same argument as in the proof of Lemma 4.4 yields

$$\left(\sum_{j\in\mathbb{Z}}|\phi_j(x)|^2\right)^{\frac{1}{2}} \lesssim_n |x|^{-n},$$

we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\phi_j * f_j^1(x)| &\lesssim \int_{|y-x_0|>2R}|x-y|^{-n}\|\{f_j(y)\}\|_{l_2}\,dy\\ &\lesssim \sum_{i=1}^{\infty}\int_{2^iR<|y-x_0|\le 2^{i+1}R}(2^iR)^{-n}\|\{f_j(y)\}\|_{l_2}\,dy\\ &\lesssim (2^iR)^{-n}|B(2^{i+1}R)|^{1-\frac{1}{p}}\big\|\|\{f_j\}\|_{l_2}\big\|_{M_{p,q}}\\ &\lesssim \sum_{i=1}^{\infty}2^{-\frac{in}{p}}|B(R)|^{-\frac{1}{p}}\|\{f_j\}\|_{M_{p,q}(l_2)}.\end{aligned}$$

Hence

$$\|T(\{f_j^1\})\|_{L_q(B(R))} \lesssim |B(R)|^{-\frac{1}{p}+\frac{1}{q}}\|\{f_j\}\|_{M_{p,q}(l_2)}.$$

Combining the estimates for $\{f_j^0\}$ and $\{f_j^1\}$, we obtain the lemma. □

Proof of Theorem 4.2. Mazzucato [3] gave only an outline of the proof. We will prove this theorem in a slightly different way by Lemmas 4.5 and 4.6.

The inequality $\lesssim$ is Lemma 4.5 itself.

For the converse inequality $\gtrsim$ we consider the map $f \mapsto \sum_{j\in\mathbb{Z}}\phi_j * f$ as the composition of two maps

$$\begin{aligned}&T : M_{p,q}(\mathbb{R}^n) \to M_{p,q}(\mathbb{R}^n, l_2), && Tf := \{\phi_j * f\}_{j\in\mathbb{Z}}\\ &S : M_{p,q}(\mathbb{R}^n, l_2) \to M_{p,q}(\mathbb{R}^n), && S(\{f_j\}) := \sum_{j\in\mathbb{Z}}\phi_j * f = \sum_{j\in\mathbb{Z}}\varphi_j * (\phi_j * f)\end{aligned}$$

with $\varphi_j = \phi_{j-1} + \phi_j + \phi_{j+1}$, where the last equality follows from $\phi_j = \varphi_j * \phi_j$. Lemma 4.6 holds if ϕ_j is replaced by φ_j. Combining this fact with Lemma 4.5, we find that the map $f \mapsto F := S(Tf)$ is a bounded operator from $M_{p,q}$ to itself. Since $\hat{f}$ coincides with $\hat{F}$ in $\mathbb{R}^n \setminus \{0\}$ by $\sum_{j\in\mathbb{Z}}\hat{\phi}_j(\xi) = 1$ for $\xi \neq 0$, we know that $\hat{f} - \hat{F}$ is supported on the origin, and hence that $f - F$ is a polynomial. By Lemma 2.7 we obtain $f = F$. This completes the proof. □

5 Besov-Morrey and Triebel-Lizorkin-Morrey spaces

In this section we define various function spaces associated with the Morrey space, and show the embedding theorems.

Let ψ and $\{\phi_j\}_{j\in\mathbb{Z}}$ be as defined in (4.1) and (4.2). Remember that the space $M_{p,q}(\mathbb{R}^n, l_r)$ and the norm $\|\{f_j\}\|_{M_{p,q}(l_r)}$ are defined in (4.3). We also define $l_r(M_{p,q}(\mathbb{R}^n))$ to be the space of all l_r-valued functions $\{f_j\}_{j\in\mathbb{Z}}$ satisfying $\{\|f_j\|_{M_{p,q}}\}_{\in\mathbb{Z}} \in l_r$, and set

$$\|\{f_j\}\|_{l_r(M_{p,q})} := \left(\sum_{j\in\mathbb{Z}} (\|f_j\|_{M_{p,q}})^r\right)^{\frac{1}{r}}.$$

The space $l_r(L_p(\mathbb{R}^n))$ and the norm $\|\{f_j\}\|_{l_r(L_p)}$ are defined similarly.

Let $0 < q \leq p < \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$. The Triebel-Lizorkin-Morrey space $\mathcal{E}^s_{p,q,r}(\mathbb{R}^n)$ is the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ whose quasi-norms

$$\|f\|_{\mathcal{E}^s_{p,q,r}} := \|\psi * f\|_{M_{p,q}} + \left\| \left(\sum_{j=1}^{\infty} 2^{jsr} |\phi_j * f|^r\right)^{\frac{1}{r}} \right\|_{M_{p,q}}$$

are finite. Similarly, the homogeneous Triebel-Lizorkin-Morrey space $\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)$, the Besov-Morrey space $\mathcal{N}^s_{p,q,r}(\mathbb{R}^n)$ and the homogeneous Besov-Morrey space $\dot{\mathcal{N}}^s_{p,q,r}(\mathbb{R}^n)$ are defined by the quasi-norms

$$\|f\|_{\dot{\mathcal{E}}^s_{p,q,r}} := \|\{2^{js}\phi_j * f\}\|_{M_{p,q}(l_r)} = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsr} |\phi_j * f|^r\right)^{\frac{1}{r}} \right\|_{M_{p,q}},$$

$$\|f\|_{\mathcal{N}^s_{p,q,r}} := \|\psi * f\|_{M_{p,q}} + \left(\sum_{j=1}^{\infty} (2^{js}\|\phi_j * f\|_{M_{p,q}})^r\right)^{\frac{1}{r}},$$

$$\|f\|_{\dot{\mathcal{N}}^s_{p,q,r}} = \|\{2^{js}\phi_j * f\}\|_{l_r(M_{p,q})} := \left(\sum_{j=-\infty}^{\infty} (2^{js}\|\phi_j * f\|_{M_{p,q}})^r\right)^{\frac{1}{r}},$$

respectively. In view of $M_{p,p}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, we have the relationships of the above spaces with the Triebel-Lizorkin space $F^s_{p,r}(\mathbb{R}^n)$, the Besov space $B^s_{p,r}(\mathbb{R}^n)$, the corresponding homogeneous spaces $\dot{F}^s_{p,r}(\mathbb{R}^n)$ and $\dot{B}^s_{p,r}(\mathbb{R}^n)$:

$$F^s_{p,r}(\mathbb{R}^n) = \mathcal{E}^s_{p,p,r}(\mathbb{R}^n), \qquad B^s_{p,r}(\mathbb{R}^n) = \mathcal{N}^s_{p,p,r}(\mathbb{R}^n),$$
$$\dot{F}^s_{p,r}(\mathbb{R}^n) = \dot{\mathcal{E}}^s_{p,p,r}(\mathbb{R}^n), \qquad \dot{B}^s_{p,r}(\mathbb{R}^n) = \dot{\mathcal{N}}^s_{p,p,r}(\mathbb{R}^n).$$

Exceptionally, the Besov space $B_{\infty,r}(\mathbb{R}^n)$ can be defined for $p = \infty$ by considering $M_{\infty,\infty}(\mathbb{R}^n)$ as $L_\infty(\mathbb{R}^n)$. The special case $r = 2$ gives the definition of the Sobolev space and the homogeneous one:

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \qquad \dot{H}^s_p(\mathbb{R}^n) = \dot{F}^s_{p,2}(\mathbb{R}^n).$$

We define the powered Hardy-Littlewood maximal operator M_η by

$$M_\eta f = M[|f|^\eta]^{\frac{1}{\eta}}$$

for $\eta > 0$. The following theorem plays a key role to investigate the function spaces defined above, especially for the case $0 < q \leq p \leq 1$ or $0 < r \leq 1$.

Theorem 5.1 (Plancherel-Polya-Nikolskii's inequality)**.** *Let* $\eta > 0$, $R > 0$ *and* $f \in \mathcal{S}'(\mathbb{R}^n)$ *with* $\hat{f} \subset B(0, R)$. *Then*

$$R^{-1} \sup_{y \in \mathbb{R}^n} \frac{|\nabla f(x-y)|}{(1+R|y|)^{\frac{n}{\eta}}} \lesssim_{n,\eta} \sup_{y \in \mathbb{R}^n} \frac{|f(x-y)|}{(1+R|y|)^{\frac{n}{\eta}}} \lesssim_{n,\eta} M_\eta f(x)$$

with $M_\eta f = M[|f|^\eta]^{1/\eta}$.

Proof. We follow the proof given in [6] and [12].

Step 1. We first assume that $R = 1$. Take a C^∞ function so that $\chi_{B(1)} \le \hat{\varphi} \le \chi_{B(2)}$. Since $f = \varphi * f$, we have

$$\partial_j f(x-y) = (\partial_j \varphi) * f(x-y) = \int_{\mathbb{R}^n} \partial_j \varphi(z) f(x-y-z)\, dz.$$

The inequality $(1+|y+z|) \le (1+|y|)(1+|z|)$ gives

$$\begin{aligned} \frac{|\partial_j f(x-y)|}{(1+|y|)^{\frac{n}{\eta}}} &\le \int_{\mathbb{R}^n} |\partial_j \varphi(z)|(1+|z|)^{\frac{n}{\eta}} \cdot \frac{|f(x-y-z)|}{(1+|y+z|)^{\frac{n}{\eta}}}\, dz \\ &\le \sup_y \frac{|f(x-y)|}{(1+|y|)^{\frac{n}{\eta}}} \int_{\mathbb{R}^n} |\partial_j \varphi(z)|(1+|z|)^{\frac{n}{\eta}}\, dz. \end{aligned}$$

Thus we obtain the first inequality.

Step 2. We continue to assume that $R = 1$. Let $0 < \delta < 1$ and $x \in \mathbb{R}^n$. Assume that $|f(x)|$ attains the minimum at x_0 in the ball $B(x, \delta)$. Since $f(x) - f(x_0) = \int_0^1 (x - x_0) \cdot \nabla f((1-\theta)x_0 + \theta x)\, d\theta$, we have

$$\begin{aligned} |f(x)| &\le |f(x_0)| + |f(x) - f(x_0)| \\ &\le \left(\frac{1}{|B(\delta)|} \int_{B(x,\delta)} |f(z)|^\eta\, dz \right)^{\frac{1}{\eta}} + \delta \sup_{z \in B(x,\delta)} |\nabla f(z)|. \end{aligned}$$

Replacing x by $x - y$, we have

$$\begin{aligned} |f(x-y)| &\le \left(\frac{1}{|B(\delta)|} \int_{B(x,|y|+1)} |f(z)|^\eta\, dz \right)^{\frac{1}{\eta}} + \delta \sup_{z \in B(1)} |\nabla f(x-y-z)| \\ &\le \left(\frac{(1+|y|)}{\delta} \right)^{\frac{n}{\eta}} M_\eta f(x) + \delta \sup_{z \in B(1)} \frac{|\nabla f(x-y-z)|}{(1+|y+z|)^{\frac{n}{\eta}}} \cdot (1+|y|)^{\frac{n}{\eta}} (1+|z|)^{\frac{n}{\eta}}. \end{aligned}$$

Dividing by $(1+|y|)^{n/\eta}$ and taking the supremum, we get

$$J := \sup_y \frac{|f(x-y)|}{(1+|y|)^{\frac{n}{\eta}}} \le \delta^{-\frac{n}{\eta}} M_\eta f(x) + 2^{\frac{n}{\eta}} \delta \sup_y \frac{|\nabla f(x-y)|}{(1+|y|)^{\frac{n}{\eta}}}.$$

This combined with the first inequality yields

$$J \le \delta^{-\frac{n}{\eta}} M_\eta f(x) + C_0 \delta J$$

with C_0 depending only on n and η. Choosing δ so that $C_0 \delta = \frac{1}{2}$, we obtain the second inequality.

Step 3. We finally consider the general $R > 0$. Applying the result for $R = 1$ to the function $f_{R^{-1}}(x) := f(x/R)$, and observing $M_\eta f_{R^{-1}}(x) = M_\eta f(x/R)$ and so on, we obtain the desired result. □

Lemma 5.2. *Let* $0<\eta<q\le p<\infty$. *Then*

$$\|M_\eta f\|_{M_{p,q}} \lesssim_{n,p,q,\eta} \|f\|_{M_{p,q}}.$$

Proof. By definition

$$\|M_\eta f\|_{M_{p,q}} = \sup_B |B|^{\frac{1}{p}-\frac{1}{q}} \|M_\eta f\|_{L_q(B)} = \sup_B \left[|B|^{\frac{\eta}{p}-\frac{\eta}{q}} \left(\int_B M[|f|^\eta]^{\frac{q}{\eta}} \right)^{\frac{\eta}{q}} \right]^{\frac{1}{\eta}}.$$

Since M is bounded on $M_{p/\eta,q/\eta}(\mathbb{R}^n)$ by Theorem 3.2, we have

$$\|M_\eta f\|_{M_{p,q}} \lesssim \sup_B \left[|B|^{\frac{\eta}{p}-\frac{\eta}{q}} \left(\int_B (|f|^\eta)^{\frac{q}{\eta}} \right)^{\frac{\eta}{q}} \right]^{\frac{1}{\eta}} = \sup_B |B|^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f|^q \, dx \right)^{\frac{1}{q}},$$

which gives the boundedness of M_η. □

Lemma 5.3. *Let* $0<q\le p<\infty$ *and* $R>0$. *If* $f\in M_{p,q}(\mathbb{R}^n)$ *satisfies* $\operatorname{supp}\hat{f} \subset B(0,R)$, *then* $f\in L_\infty(\mathbb{R}^n)$ *with*

$$\|f\|_{L_\infty} \lesssim_{n,p,q} R^{\frac{n}{p}} \|f\|_{M_{p,q}}. \tag{5.1}$$

In addition, if $0<q_1\le p_1<\infty$ *and* $p/p_1=q/q_1<1$, *then* $f\in M_{p_1,q_1}(\mathbb{R}^n)$ *with*

$$\|f\|_{M_{p_1,q_1}} \lesssim_{n,p,q,p_1} R^{\frac{n}{p}-\frac{n}{p_1}} \|f\|_{M_{p,q}}. \tag{5.2}$$

Proof. Step 1. To understand the idea of the proof let us begin with the case $p=q$ and $R=1$. Assume that $f\in L_p(\mathbb{R}^n)$ satisfies $\hat{f}\subset B(0,1)$. Let B be a ball of radius 1. By Theorem 5.1 with $0<\eta<p$ we have, for $x\in B$ and $y\in B$,

$$|f(x)| = \sup_{z\in B(0,2)} |f(y-z)| \le \sup_{y\in B(0,2)} \frac{3^{\frac{n}{\eta}}|f(y-z)|}{(1+|z|)^{\frac{n}{\eta}}} \lesssim M_\eta f(y).$$

Taking the $L_p(B)$ norms as a function of y, we get

$$|B|^{\frac{1}{p}} \|f\|_{L_\infty(B)} \lesssim \|M_\eta f\|_{L_p(B)}.$$

Since the maximal operator is bounded on $L_{p/\eta}$ by Theorem 3.1, we get $\|f\|_{L_\infty(B)} \lesssim \|f\|_{L_p}$. Taking the supremum with B ranging over all balls of radius 1, we obtain (5.1) for $p=q$ and $R=1$.

Step 2. We next assume that $f\in M_{p,q}(\mathbb{R}^n)$ satisfies $\hat{f}\subset B(0,1)$. Replacing the $L_p(B)$ by $L_q(B)$ in the argument of Step 1, and choosing η so that $0<\eta<q\le p<\infty$, we have

$$|B|^{\frac{1}{q}} \|f\|_{L_\infty(B)} \lesssim \|M_\eta f\|_{L_q(B)} \le |B|^{-\frac{1}{p}+\frac{1}{q}} \|M_\eta f\|_{p,q} \lesssim \|f\|_{p,q},$$

where the last inequality follows by Lemma 5.2. Taking the supremum with B ranging over all balls of radius 1, we obtain (5.1) with $R=1$.

Step 3. We consider the general R by the scaling $f_{R^{-1}}(x)=f(x/R)$. Since $\mathcal{F}f_{R^{-1}}(\xi)=R^n\hat{f}(R\xi)$ is supported on $B(0,1)$, we have

$$\|f\|_{L_\infty} = \|f_{R^{-1}}\|_{L_\infty} \lesssim \|f_{R^{-1}}\|_{p,q} = R^{\frac{n}{p}}\|f\|_{p,q},$$

where the last equality follows by Lemma 2.1.

Step 4. Inequality (5.2) follows from (5.1). Indeed, we have

$$\|f\|_{L_{q_1}(B)} \leq \|f\|_{L_\infty}^{1-\frac{q}{q_1}} \|f\|_{L_q(B)}^{\frac{q}{q_1}} \lesssim (R^{\frac{n}{p}}\|f\|_{p,q})^{1-\frac{q}{q_1}} \left(|B|^{-\frac{1}{p}+\frac{1}{q}}\|f\|_{p,q}\right)^{\frac{q}{q_1}},$$

which yields (5.2) by $q/q_1 = p/p_1$. □

Theorem 5.4. *(i) Let* $0 < q \leq p < \infty$, $0 < r \leq \infty$ *and* $s \in \mathbb{R}$. *Then*

$$\mathcal{E}^s_{p,q,r}(\mathbb{R}^n) \subset \mathcal{E}^s_{p,q,\infty}(\mathbb{R}^n) \subset B^{s-\frac{n}{p}}_{\infty,\infty}(\mathbb{R}^n).$$

(ii) Let $0 < q \leq p < \infty$, $0 < r \leq \infty$ *and* $s \in \mathbb{R}$. *Then*

$$\mathcal{N}^s_{p,q,r}(\mathbb{R}^n) \subset B^{s-\frac{n}{p}}_{\infty,r}(\mathbb{R}^n).$$

(iii) Let $0 < p < p_1 < \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ *and* $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. *Then*

$$F^s_{p,\infty}(\mathbb{R}^n) \subset F^{s_1}_{p_1,r}(\mathbb{R}^n).$$

(iv) Let $0 < q \leq p < p_1 < \infty$, $p/p_1 = q/q_1 (< 1)$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ *and* $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. Then

$$\mathcal{E}^s_{p,q,\infty}(\mathbb{R}^n) \subset \mathcal{E}^{s_1}_{p_1,q_1,r}(\mathbb{R}^n).$$

(v) Let $0 < p < p_1 \leq \infty$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ *and* $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. *Then*

$$B^s_{p,r}(\mathbb{R}^n) \subset B^{s_1}_{p_1,r}(\mathbb{R}^n).$$

(vi) Let $0 < q \leq p < p_1 < \infty$, $p/p_1 = q/q_1 (< 1)$, $0 < r \leq \infty$, $s \in \mathbb{R}$, $s_1 \in \mathbb{R}$ *and* $s - \frac{n}{p} = s_1 - \frac{n}{p_1}$. *Then*

$$\mathcal{N}^s_{p,q,r}(\mathbb{R}^n) \subset \mathcal{N}^{s_1}_{p_1,q_1,r}(\mathbb{R}^n).$$

Remark. The results related to Morrey spaces in Theorem 5.4 are obtained by Sawano-Tanaka [9], Sawano [5] and Sawano-Sugano-Tanaka [8] (see also [10]).

Proof. In the proof we only estimate $\phi_j * f$, since the estimate for $\psi * f$ can be dealt with similarly.

(i) The first inclusion follows from $l_r \subset l_\infty$. Let $f \in \mathcal{E}^s_{p,q,\infty}$ and set $G = \sup_{j\in\mathbb{N}} 2^{js}|\phi_j * f|$. Clearly $|\phi_j * f| \leq 2^{-js}G$. By Lemma 5.3 we have

$$\|\phi_j * f\|_\infty \lesssim 2^{\frac{jn}{p}}\|\phi_j * f\|_{p,q} \leq 2^{j(\frac{n}{p}-s)}\|G\|_{p,q},$$

which implies

$$\sup_{j\in\mathbb{N}} 2^{j(s-\frac{n}{p})}\|\phi_j * f\|_\infty \leq \|f\|_{\mathcal{E}^s_{p,q,\infty}}.$$

(ii) From Lemma 5.3 it follows that

$$2^{j(s-\frac{n}{p})}\|\phi_j * f\|_\infty \lesssim 2^{j(s-\frac{n}{p})} \cdot 2^{\frac{jn}{p}}\|\phi_j * f\|_{p,q} = 2^{js}\|\phi_j * f\|_{p,q},$$

which gives $\|f\|_{B^{s-n/p}_{\infty,r}} \lesssim \|f\|_{\mathcal{N}^s_{p,q,r}}$.

(iii) Let $f \in F^s_{p,\infty}$ and set $G = \sup_{j\in\mathbb{N}} 2^{js}|\phi_j * f|$. Clearly $|\phi_j * f| \le 2^{-js}G$. By Lemma 5.3 we have

$$\|\phi_j * f\|_\infty \lesssim 2^{\frac{jn}{p}}\|\phi_j * f\|_p \le 2^{j(\frac{n}{p}-s)}\|G\|_p.$$

Thus

$$2^{js_1}|\phi_j * f| \lesssim \min\{2^{-j(s-s_1)}G, 2^{j(s_1-s+\frac{n}{p})}\|G\|_p\} = \min\{2^{-j(\frac{n}{p}-\frac{n}{p_1})}G, 2^{j\cdot\frac{n}{p_1}}\|G\|_p\}.$$

Applying Lemma 5.5 below with $a = nr(\frac{1}{p}-\frac{1}{p_1})$, $b = nr\cdot\frac{1}{p_1}$, $A = G^r$, and $B = \|G\|_p^r$, we have

$$\left(\sum_j 2^{js_1r}|\phi_j * f|^r\right)^{\frac{1}{r}} \lesssim G^{\frac{p}{p_1}}\|G\|_p^{1-\frac{p}{p_1}}.$$

Taking the L_{p_1} norm yields $\|f\|_{F^{s_1}_{p_1,r}} \lesssim \|G\|_p$.

(iv) We can proceed in the same way as in (iii) by replacing the L_p norm by the $M_{p,q}$ norm. Thus

$$H := \left(\sum_j 2^{js_1r}|\phi_j * f|^r\right)^{\frac{1}{r}} \lesssim G^{\frac{p}{p_1}}\|G\|_{p,q}^{1-\frac{p}{p_1}} = G^{\frac{q}{q_1}}\|G\|_{p,q}^{1-\frac{q}{q_1}}.$$

Hence, with the assumption $p_1/p = q_1/q$,

$$\|H\|_{L_{q_1}(B)} \le \|G\|_{L_q(B)}^{\frac{q}{q_1}}\|G\|_{p,q}^{1-\frac{q}{q_1}} \le (|B|^{-\frac{1}{p}+\frac{1}{q}})^{\frac{q}{q_1}}\|G\|_{p,q} = |B|^{-\frac{1}{p_1}+\frac{1}{q_1}}\|G\|_{M_{p,q}},$$

which implies (iv).

(v) By Lemma 5.3 we have

$$2^{js_1}\|\phi_j * f\|_{p_1} \lesssim 2^{js_1}\cdot 2^{j(\frac{n}{p}-\frac{n}{p_1})}\|\phi_j * f\|_p = 2^{js}\|\phi_j * f\|_p,$$

which gives (v).

Item (vi) can be proved in the same way as (v), if we replace the L_p norm by the $M_{p,q}$ norm. □

Lemma 5.5. *Let a, b, A, B be positive numbers. Then*

$$\sum_{j\in\mathbb{Z}} \min\{2^{-ja}A, 2^{jb}B\} \le \frac{2^{\min\{a,b\}}}{\log 2}\left(\frac{1}{a}+\frac{1}{b}\right)A^{\frac{b}{a+b}}B^{\frac{a}{a+b}}.$$

Proof. We compare the sum in the lemma with

$$J := \int_0^\infty \min\{At^{-a}, Bt^b\}\,\frac{dt}{t}.$$

Decomposing $(0, \infty)$ into the union of $(2^j, 2^{j+1})$ with $j\in\mathbb{Z}$, we have

$$J = \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \min\{At^{-a}, Bt^b\}\,\frac{dt}{t} \ge \sum_{j\in\mathbb{Z}} (\log 2)2^{-a}\min\{A2^{-aj}, B2^{bj}\}.$$

On the other hand, changing the variables $t = cs$ and taking c so that $Ac^{-a} = Bc^{b}$, we have

$$J = Ac^{-a}\int_0^\infty \min\{s^{-a}, s^b\}\,\frac{ds}{s} = A^{\frac{b}{a+b}}B^{\frac{a}{a+b}}\left(\frac{1}{a}+\frac{1}{b}\right).$$

Since replacement of j by $-j$ changes the roles of a, A and those of b, B, we obtain the desired inequality. $\square$

6 Lifting property

As usual we define the seminorms in $\mathcal{S}(\mathbb{R}^n)$; for $N \in \mathbb{N}$ we set

$$q_N(f) = \max_{|\alpha|\le N}\max_{|\beta|\le N}\sup_{x\in\mathbb{R}^n}|x^\beta\partial^\alpha f(x)|.$$

For $H \in \mathcal{S}(\mathbb{R}^n)$ we define the operator $H(D) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$H(D)f = \mathcal{F}^{-1}(H\mathcal{F}^{-1}f). \tag{6.1}$$

Lemma 6.1. *Let $0 < q \le p < \infty$ and $0 < r \le \infty$. Assume that a series $\{f_j\}_{j\in\mathbb{Z}}$ in $\mathcal{S}'(\mathbb{R}^n)$ satisfies $\{f_j\} \in M_{p,q}(\mathbb{R}^n, l_r)$ and*

$$\operatorname{supp}\hat{f}_j \subset \{\xi : |\xi| \le 2^j\},$$

and that $\{H_j\}_{j\in\mathbb{Z}}$ in $\mathcal{S}(\mathbb{R}^n)$ satisfies

$$\sup_{j\in\mathbb{Z}} q_N(H_j(2^j\,\cdot)) < \infty$$

for each $N \in \mathbb{N}_0$. Then $\{H_j(D)f_j\} \in M_{p,q}(\mathbb{R}^n, l_r)$ with

$$\|\{H_j(D)f_j\}\|_{M_{p,q}(l_r)} \lesssim_{n,p,q,r} \left(\sup_{j\in\mathbb{Z}} q_N(H_j(2^j\,\cdot))\right)\|\{f_j\}\|_{M_{p,q}(l_r)}$$

with $N = [n/\eta] + n + 2$.

Remark. This lemma also holds if $\mathbb{Z}$ is replaced by $\mathbb{N}$, since we may apply the lemma to the case where $f_j = 0$ for $j \le 0$.

The statement of Lemma 6.1 also holds if we replace $M_{p,q}(\mathbb{R}^n, l_r)$ by $l_r(M_{p,q}(\mathbb{R}^n))$; the proof is based on (6.2) below.

Proof. Let $\eta = \frac{1}{2}\min\{q, r\}$. By Theorem 5.1 we have

$$\begin{aligned}|H_j(D)f_j(x)| &\le \int_{\mathbb{R}^n}|\mathcal{F}^{-1}H_j(y)|(1+2^j|y|)^{\frac{n}{\eta}}\frac{|f_j(x-y)|}{(1+2^j|y|)^{\frac{n}{\eta}}}\,dy\\ &\lesssim M_\eta[f_j](x)\int_{\mathbb{R}^n}2^{-jn}|\mathcal{F}^{-1}H_j(2^{-j}y)|(1+|y|)^{\frac{n}{\eta}}\,dy.\end{aligned}$$

Choose N so that $N \ge n+1$ and $N - \frac{n}{\eta} > n$. Then

$$
\begin{aligned}
2^{-jn}|\mathcal{F}^{-1}H_j(2^{-j}y)|(1+|y|)^N &\le C_{n,N}\sum_{|\alpha|\le N} 2^{-jn}|y^\alpha \mathcal{F}^{-1}H_j(2^{-j}y)| \\
&\le C_{n,N}\sum_{|\alpha|\le N}(2\pi)^{|\alpha|}\int_{\mathbb{R}^n}|\partial_\xi^\alpha\{H_j(2^j\xi)\}|\,d\xi \\
&\le C'_{n,N}q_N(H_j(2^j\,\cdot\,))\int_{\mathbb{R}^n}(1+|\xi|)^{-n-1}\,d\xi.
\end{aligned}
$$

Combining the above estimates, we have

$$
|H_j(D)f_j(x)| \lesssim q_N(H_j(2^j\,\cdot\,))M_\eta[f_j](x). \tag{6.2}
$$

As we proved Lemma 5.2, we can show that Theorem 3.4 also holds if M is replaced by the powered Hardy-Littlewood operator M_η. Hence the lemma follows from (6.2). □

Theorem 6.2. *Let* $0<q\le p<\infty$, $0<r\le\infty$, $s\in\mathbb{R}$ *and* $\alpha\in\mathbb{N}_0^n$. *Then the following linear operators are bounded*:

$$
D^\alpha:\mathcal{E}^s_{p,q,r}(\mathbb{R}^n)\to\mathcal{E}^{s-|\alpha|}_{p,q,r}(\mathbb{R}^n),\qquad D^\alpha:\dot{\mathcal{E}}^s_{p,q,r}(\mathbb{R}^n)\to\dot{\mathcal{E}}^{s-|\alpha|}_{p,q,r}(\mathbb{R}^n),
$$
$$
D^\alpha:\mathcal{N}^s_{p,q,r}(\mathbb{R}^n)\to\mathcal{N}^{s-|\alpha|}_{p,q,r}(\mathbb{R}^n),\qquad D^\alpha:\dot{\mathcal{N}}^s_{p,q,r}(\mathbb{R}^n)\to\dot{\mathcal{N}}^{s-|\alpha|}_{p,q,r}(\mathbb{R}^n).
$$

Remark. A Generalization of this theorem is found in Sawano-Tanaka [9].

Proof. We give the proof only for the non-homogeneous Triebel-Lizorkin spaces; we can deal with the other spaces similarly. Set $\varphi_j=\phi_{j-1}+\phi_j+\phi_{j+1}$. Then

$$
2^{j(s-|\alpha|)}\phi_j*(D^\alpha f)=2^{j(s-|\alpha|)}\varphi_j*\phi_j*(D^\alpha f)=H_j(D)[2^{sj}\phi_j*f]
$$

with $H_j(\xi)=(2\pi\xi/2^j)^\alpha\hat{\varphi}_0(\xi/2^j)$. Since $H_j(2^j\xi)=(2\pi\xi)^\alpha\hat{\phi}_0(\xi)$, which is independent of j, we conclude that D^α is bounded by Lemma 6.1. □

We say that $a(\xi)$ is a symbol of order m if $a(\xi)$ satisfies

$$
|\partial_\xi^\alpha a(\xi)|\le C_\alpha\langle\xi\rangle^{m-|\alpha|}
$$

for every $\alpha\in\mathbb{N}_0^n$. A typical example of $a(\xi)$ is $\langle\xi\rangle^m$ with $\langle\xi\rangle=(1+|\xi|^2)^{\frac{1}{2}}$. We note that the operator $a(D)$ can be defined as in (6.1), since $g\mapsto ag$ is a bounded operator on $\mathcal{S}(\mathbb{R}^n)$.

Theorem 6.3. *Let* $0<q\le p<\infty$, $0<r\le\infty$ *and* $s\in\mathbb{R}$. *If* $a(\xi)$ *is a symbol of order* m, *then the operators*

$$
a(D):\ \mathcal{E}^s_{p,q,r}(\mathbb{R}^n)\to\mathcal{E}^{s-m}_{p,q,r}(\mathbb{R}^n),\qquad a(D):\ \mathcal{N}^s_{p,q,r}(\mathbb{R}^n)\to\mathcal{N}^{s-m}_{p,q,r}(\mathbb{R}^n)
$$

are bounded.

Proof. We write

$$
2^{j(s-m)}\phi_j*(a(D)f)=2^{j(s-m)}\varphi_j*\phi_j*(a(D)f)=H_j(D)(2^{js}\phi_j*f)
$$

with $H_j(\xi) = 2^{-mj}a(\xi)\varphi_j(\xi)$. Since the derivatives of $H_j(2^j\xi) = 2^{-mj}a(2^j\xi)\varphi_0(\xi)$ is given by

$$\partial_\xi^\alpha(H_j(2^j\xi)) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}2^{(|\beta|-m)j}(\partial^\beta a)(2^j\xi)(\partial^{\alpha-\beta}\varphi_0)(\xi),$$

and they are supported on $\frac{1}{4} \le |\xi| \le 4$, we have

$$|\partial_\xi^\alpha(H_j(2^j\xi))| \lesssim \sum_{\beta\le\alpha}2^{(|\beta|-m)j}\langle 2^j\xi\rangle^{m-|\beta|} \lesssim 1,$$

since $2^j|\xi| \le \langle 2^j\xi\rangle \le \{(2\cdot 2^j|\xi|)^2+(2^j|\xi|)^2\}^{\frac{1}{2}} \le 3\cdot 2^j|\xi|$. Thus we conclude the boundedness of $a(D)$ by applying Lemma 6.1 for the Triebel-Lizorkin space and the remark of Lemma 6.1 for the Besov space.

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