Working Paper Series

No. 24-02

March 2025

Adjustment Process in a Cournot Oligopoly Model: Global Stability and Chaotic Motion

Hiroyuki Yoshida

Giuseppe Orlando

Research Institute of Economic Science

College of Economics, Nihon University

Adjustment Process in a Cournot Oligopoly Model: Global Stability and Chaotic Motion

Hiroyuki Yoshida*
College of Economics, Nihon University.

Giuseppe Orlando†
Department of Economics and Finance, University of Bari.

March 30, 2025

Abstract

We have investigated the dynamic nature of the adjustment process in a Cournot model with linear demand and constant marginal costs. In particular, we considered two types of models. First, we formulated a dynamic adjustment process without time lags. In this case we established the local and the global stability of the Cournot-Nash equilibrium. Second, we examined a dynamic adjustment process with four time lags. In this process we confirmed that irregular and chaotic fluctuations could emerge in the duopoly market.

*E-mail address: yoshida.hiroyuki@nihon-u.ac.jp. †E-mail address: giuseppe.orlando@uniba.it.

1 Introduction

In his seminal work, Cournot (1838, Chapter 7) constructed the theory of oligopoly in a mathematical framework. His model captured the strategic interactions among a small number of firms in a market for a single homogeneous commodity; each firm tries to maximize its profit by taking the output choices of the other firms as given. The equilibrium point in his model is defined as the intersection of the reaction functions. Such an equilibrium notion is in essence identical with the Nash equilibrium concept. In this sense Cournot anticipated Nash (1951) more than a century ago. There is no disagreement on the point that Cournot's oligopoly model is the classic instance of a noncooperative game in economics.

In addition to the development of the new equilibrium concept, he formulated an adjustment process for his duopoly model. By using a graphical argument he showed that the equilibrium point is stable under the assumption of monotone reaction functions. His graphical analysis aroused considerable interest in the stability problem and yielded a large number of papers; see, for instance, Theocharis (1960), Fischer (1961), and Hahn (1962). Furthermore, Okuguchi (1976) provides a comprehensive review of oligopoly models.

More recently, the development of chaos theory in discrete-time dynamical systems has stimulated the reconsideration on the dynamic nature of the Cournot model. In the pioneering paper Rand (1978) rigorously showed the emergence of cyclical and chaotic dynamics in the discrete-time adjustment process when unimodal reaction functions are considered. Following Rand, Puu (1991) showed that reaction functions are unimodal by assuming linear cost functions and a hyperbolic demand function. Moreover, Kopel (1996) obtained unimodal reaction functions by introducing nonlinear cost functions and a linear demand function. These papers are closely related to each other in that they used coupled oscillators in discrete dynamical systems to demonstrate the occurrence of complex dynamics. These papers have been further developed by many authors. See, for example, Bischi and Kopel (2001), Bischi and Lamantia (2002). Further contributions were presented in Puu and Sushko (2002).

Unfortunately, however, very little attention has been paid to the investigation of the complex duopoly dynamics in continuous-time models so far. The reason is due to a lack of analytical methods for establishing the existence of continuous chaos, while it is well known that specific dynamical systems, such as the Lorenz system (1963) and the Rössler system (1976), display chaotic behavior for certain values of system parameters. The Lorenz system is described by a three-dimensional system of ordinary differential equations representing fluid convection in the atmosphere. Moreover, the Rössler system is also expressed by a three-dimensional system of ordinary differential equations, which is applied to the analysis of chemical reactions. It is astonishing that even these simple and deterministic systems could exhibit complex and unpredictable behaviors in the continuous-time scale.

The purpose of the present paper is twofold. The first aim is to formulate a basic model of the Cournot adjustment process and establish the global stability of the Cournot-Nash equilibrium by using a system of ordinary differential equations.

The second aim is to investigate the possibility of persistent fluctuations in output within the Cournot adjustment process under continuous time. Figure 1 illustrates the evolution of market shares in the global smartphone market. At a glance, it is evident that the market share of each company fluctuates. Additionally, it is also apparent that the market rankings occasionally change. For example, Samsung held a market share of just 2.3% in April 2010, but this figure increased steadily over time. By August 2023, Samsung had surpassed a 25% share, becoming the market leader. However, since March 2023, Apple has taken the lead, with Samsung falling to second place. These observations highlight that market shares fluctuate over time. Such dynamics

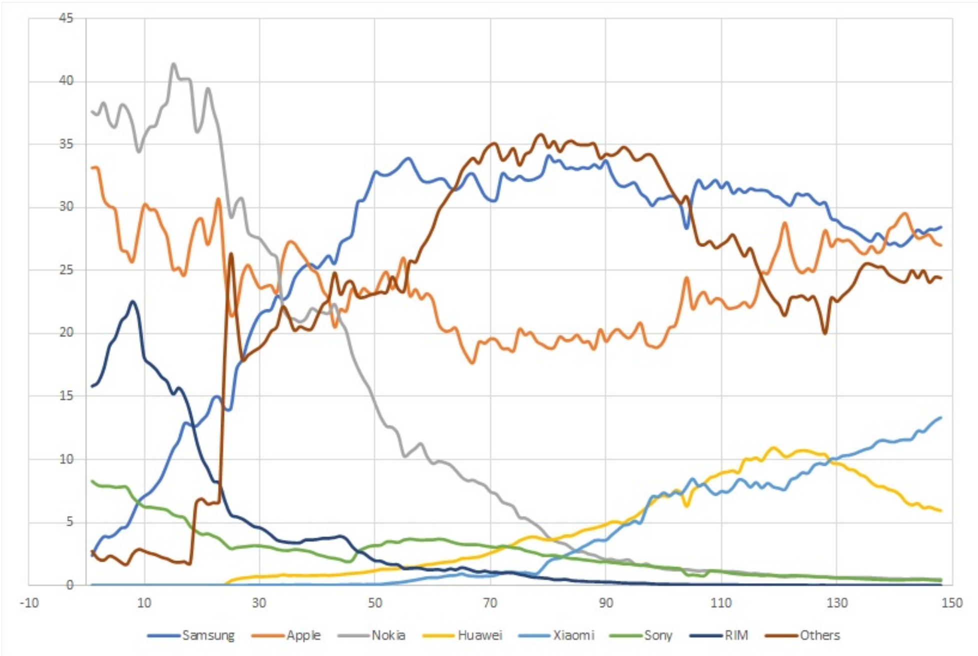

Figure 1: The evolution of market shares in the global smartphone market

are not unique to the mobile device market; similar patterns can also be observed in other industries, such as automobiles and cigarette. On this point, see Mazzucato and Semmler (1999) and Gallet and List (2001).

The present paper shares the same theoretical orientation as the work of Furth (2009). In his paper, Furth provides an important and clear result: no cycles appear in the homogeneous Cournot oligopoly model with quantity adjustments. In contrast to Furth, however, we explore the emergence of limit cycles and chaotic attractors in homogeneous Cournot duopoly. Since his analysis is completely correct in the mathematical context, for our purpose, we introduce two 'alternative' elements from the economic point of view. The elements are discrete time lags and a growth rate rule for the adjustment process. These elements yield a system of nonlinear delay differential equations, which is essentially identical with a model of Shibata and Saito (1980). They investigate the population dynamics of two competing species with fixed lags and showed the appearance of limit cycles and chaotic attractors by means of numerical simulations. Their model is a significant contribution to the field of mathematical biology.

Finally, we make a remark on the modeling of discrete time lags in the Cournot model. It has already been studied by Howroyd and Russell (1984) and Russell, Rickard, and Howroyd (1986).[1] They derive analytical results on the stability conditions by investigating the systems of linear delay differential equations. The economic interpretation of their results suggests that the equilibrium point is stable when the delays are small. By means of numerical simulations we can obtain similar results in our model. However, our model differs from theirs in that we can observe numerically periodic cycles and chaotic attractors in our system. This is because we deal with the system of nonlinear delay differential equations, while they examine the linear systems.

2 The basic framework

We consider a Cournot duopoly model. Two firms produce a homogeneous product with output levels x_1 and x_2. Let the market demand function be given by

$$p = a - b(x_1 + x_2),\ \ a > 0, b > 0, \tag{1}$$

where p is the price of output. Assuming constant marginal cost c_i for firm i $(i = 1, 2)$, we obtain the profit functions of the two firms:

$$\pi_1 = [a - b(x_1 + x_2)]x_1 - c_1 x_1, \tag{2}$$

$$\pi_2 = [a - b(x_1 + x_2)]x_2 - c_2 x_2. \tag{3}$$

As usual we assume that each firm maximizes its profit taking the quantity supplied by the opponent as given. Consequently, the profit maximization conditions result in

[1] Chiarella and Khomin (1996) also examine the stability properties of the Cournot oligopoly model by using a distributed lag model, which is a generalization of the discrete lag model. They apply the Hopf bifurcation theorem to study the birth of limit cycles from the Cournot-Nash equilibrium point.

the following best-reaction functions:

$$x_1 = R_1(x_2) = -\frac{1}{2}x_2 + \frac{a - c_1}{2b} \tag{4}$$

$$x_2 = R_2(x_1) = -\frac{1}{2}x_1 + \frac{a - c_2}{2b} \tag{5}$$

These equations clearly show that an optimal quantity choice for firm i depends on the output level of the opponent. From (4) and (5), we find a unique Cournot-Nash equilibrium, which corresponds to the intersection point for the reaction curves:

$$x_1^* = \frac{a - 2c_1 + c_2}{3b}, \;\; x_2^* = \frac{a + c_1 - 2c_2}{3b}. \tag{6}$$

In order to ensure positive values of x_1^* and y_2^* we require the following:

Assumption 1. $a - 2c_1 + c_2 > 0$ and $a + c_1 - 2c_2 > 0$.

In studying the dynamic duopoly model, we assume that each firm controls the growth rate of its output according to the difference between its profit maximizing output and its actual output:

$$\frac{\dot{x}_1}{x_1} = \alpha_1(R_1(x_2) - x_1), \;\; \alpha_1 > 0, \tag{7a}$$

$$\frac{\dot{x}_2}{x_2} = \alpha_2(R_2(x_1) - x_2), \;\; \alpha_2 > 0 \tag{7b}$$

where $\alpha_i (i = 1, 2)$ indicates an adjustment speed parameter, which is a positive constant.

With this assumption and the reaction functions given by (4) and (5), we obtain a nonlinear dynamical system of the form:

$$\dot{x}_1 = \alpha_1 \left(-x_1 - \frac{1}{2}x_2 + \frac{a - c_1}{2b} \right) x_1 \tag{8a}$$

$$\dot{x}_2 = \alpha_2 \left(-\frac{1}{2}x_1 - x_2 + \frac{a - c_2}{2b} \right) x_2 \tag{8b}$$

Analysis of the dynamical system (8) yields the phase diagram shown in Figure 2. The slope of the reaction function of firm 1 is steeper than that of the reaction function of firm 2. Notice that $a - c_2 > (1/2)(a - c_1)$ and $a - c_1 > (1/2)(a - c_2)$ by virtue of Assumption 1. Thus the reaction function of firm 1 must cut the reaction function of firm 2 from above at the Cournot-Nash equilibrium.

Two reaction functions divide the positive orthant of the (x_1, x_2)-plane into four regions. The actual motion of the system can be approximated by the arrows of motion. This diagram tells us that no matter where the initial output vector lies within R^2_{++}, all trajectories move toward the Cournot-Nash equilibrium point.

To find the stationary points of the above system, we solve $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ simultaneously. Then we obtain four stationary points: $E_0(0, 0), E_1((a-c_1)/(2b), 0), E_2(0, (a-c_2)/(2b))$, and $E_3(x_1^*, x_2^*)$. Note that while point E_3 is a Cournot-Nash equilibrium, the remaining three points do not satisfy the definition of Nash equilibrium. At least one firm has an incentive to change its output decision on these non-Nash equilibrium points.

We now turn to the local stability analysis of the four stationary points in turn. The local stability of a stationary point is determined by the characteristic roots of the Jacobian matrix evaluated at the point.

(i) $E_0(0, 0)$: We obtain the Jacobian matrix evaluated at E_0,

$$J_0 = \begin{bmatrix} \alpha_1(a - c_1)/(2b) & 0 \\ 0 & \alpha_2(a - c_2)/(2b) \end{bmatrix}. \tag{9}$$

Since J_0 is a diagonal matrix, it is easily checked that the characteristic roots of J_0 are $\lambda_1 = \alpha_1(a - c_1)/(2b) > 0$ and $\lambda_2 = \alpha_2(a - c_2)/(2b) > 0$. This means that E_0 is locally unstable.

(ii) $E_1((a - c_X)/(2b), 0)$: The Jacobian matrix evaluated at E_1 is

$$J_1 = \begin{bmatrix} -\alpha_1(a - c_1)/(2b) & -(1/2)\alpha_1(a - c_1)/(2b) \\ 0 & \alpha_2(a - c_2)/(2b) \end{bmatrix}. \tag{10}$$

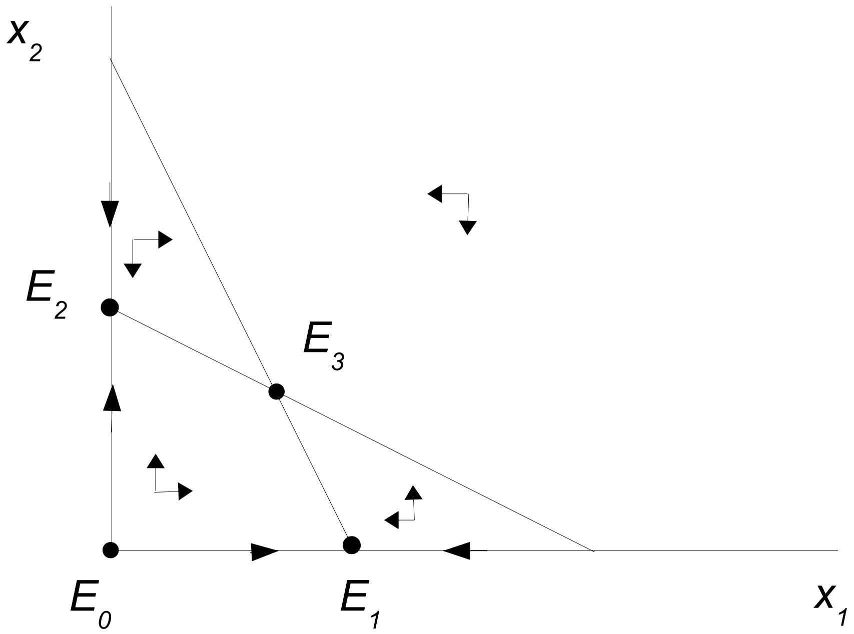

Figure 2: The phase diagram of the basic model

Since the determinant of the Jacobian matrix is negative, the matrix has two real distinct characteristic roots of opposite signs. Therefore E_1 is a saddle point. The motion is always away from point E_1 except for the stable arm of the saddle. In this case the positive part of the x_1-axis itself is the stable arm. If the initial point lies on the x_1-axis, the trajectory approaches point E_1 as $t \to +\infty$.

(iii) $E_2(0, (a - c_2)/(2b))$: In the same way, we can show that E_2 is a saddle point. The motion is always away from point E_2 except for the stable arm of the saddle. In this case the positive part of the x_2-axis itself is the stable arm. If the initial point lies on the x_2-axis, the trajectory approaches point E_2 as $t \to +\infty$.

(iv) $E_3(x_1^*, x_2^*)$: To make sure of the stability properties of the system we evaluate the Jacobian matrix at the Cournot-Nash equilibrium

$$J = \begin{bmatrix} -\alpha_1 x_1^* & -(1/2)\alpha_1 x_1^* \\ -(1/2)\alpha_2 x_2^* & -\alpha_2 x_2^* \end{bmatrix}, \tag{11}$$

and this yields the characteristic equation

$$P(\lambda) = \lambda^2 + b_1\lambda + b_2 = 0, \tag{12}$$

where,

$$b_1 = \alpha_1 x_1 + \alpha_2 x_2, \tag{13}$$

$$b_2 = (3/4)\alpha_1 x_1 \alpha_2 x_2. \tag{14}$$

From the above discussion, we can conclude that the Routh-Hurwitz conditions $(b_1 > 0, b_2 > 0)$, which are the necessary and sufficient conditions for local asymptotic stability, are always satisfied. Thus we have the following:

Proposition 1 *The steady state is locally asymptotically stable for any* $\alpha_1, \alpha_2 > 0$.

In the remainder of this section, we show the global stability of the Cournot-Nash equilibrium by using Lyapunov's second method, which is more rigorous than the phase diagram analysis. Consider a dynamical system $\dot{x} = f(x), x \in R^n$. A vector x^* is a stationary point of the dynamical system if and only if $f(x^*) = 0$. Lyapunov's second theorem on stability asserts that x^* is asymptotically stable if a scalar function $V(x)$ defined on some subset $W \subset R^n$ satisfies the following conditions

(i) $V(x^*) = 0$ and $V(x) > 0$ for $x \neq x^*$,

(ii) $\dot{V}(x) < 0$ for all $x \in W - \{x^*\}$.

The theorem provides a strict and useful tool to study global stability.[2] In fact, we can obtain the following proposition:

Proposition 2 *The Cournot-Nash equilibrium is globally asymptotically stable in* R^2_{++}.

Proof: Consider the following candidate for the Lyapunov function:

[2] For more details, see Guckenheimer and Holmes (1997) or Perko (1996).

$$V(x_1, x_2) = (x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 + (x_1 - x_1^*)(x_2 - x_2^*). \tag{15}$$

We shall check whether the above function satisfies two conditions concerning the Lyapunov function.

First, notice that (15) can be transformed into

$$V(x_1, x_2) = \frac{3}{2}\left[\frac{1}{\sqrt{2}}(x_1 - x_1^*) + \frac{1}{\sqrt{2}}(x_2 - x_2^*)\right]^2 + \frac{1}{2}\left[\frac{1}{\sqrt{2}}(x_1 - x_1^*) - \frac{1}{\sqrt{2}}(x_2 - x_2^*)\right]^2. \tag{16}$$

Accordingly, we can recognize that $V(x_1^*, x_2^*) = 0$ and $V(x_1, x_2) > 0$ for $(x_1, x_2) \neq (x_1^*, x_2^*)$.

Second, by differentiating (15) with respect to time, we can obtain

$$\dot{V}(x_1, x_2) = \dot{x}_1[2(x_1 - x_1^*) + (x_2 - x_2^*)] + \dot{x}_2[(x_1 - x_1^*) + 2(x_2 - x_2^*)]. \tag{17}$$

Note that (8) can be rearranged to

$$2\dot{x}_1 = -\alpha_1[2(x_1 - x_1^*) + (x_2 - x_2^*)]x_1 \tag{18a}$$

$$2\dot{x}_2 = -\alpha_2[(x_1 - x_1^*) + 2(x_2 - x_2^*)]x_2. \tag{18b}$$

Thus, substituting (18) into (17) gives

$$\dot{V} = -\frac{2(\dot{x}_1)^2}{\alpha_1 x_1} - \frac{2(\dot{x}_2)^2}{\alpha_2 x_2} < 0 \text{ for all } (x_1, x_2) \in R^2_{++} - \{x^*\}. \tag{19}$$

From the above results, we can conclude that (15) is a suitable Lyapunov function. In addition to this, we can choose $W = R^2_{++}$. This completes the proof.

Remark 1 When analysing the Cournot adjustment process, it is usual to assume an incremental-change rule, which can be specified as follows

$$\dot{x}_1 = \tilde{\alpha}_1(R_1(x_2) - x_1), \tag{20}$$

$$\dot{x}_2 = \tilde{\alpha}_2(R_2(x_1) - x_2). \tag{21}$$

Nevertheless, we have adopted a growth-rate rule, specified in (7). The reason why we use the growth-rate rule is that it has an important advantage over the incremental-change rule from an economic point of view: non-negative restrictions on the output levels are automatically satisfied for all $t > 0$ under the growth-rate rule. To verify this property we integrate (7a) and (7b) with respect to t, and hence obtain the following expressions:

$$x_1(t) = x_1(0) \exp\left[\int_0^t (R_1(x_2(s)) - x_1(s))ds\right], \tag{22}$$

$$x_2(t) = x_2(0) \exp\left[\int_0^t (R_2(x_1(s)) - x_2(s))ds\right]. \tag{23}$$

These equations prove clearly that the non-negativity of $x_1(t)$ and $x_2(t)$ is fulfilled for all $t > 0$ as long as initial conditions $(x_1(0), x_2(0))$ are non-negative.

Remark 2 It should be noted that the growth-rate rule is equivalent to the incremental-change rule if we assume that the adjustment speed parameters considered in (20) and (21) are linear functions such that $\tilde{\alpha}_1 = v_1 x_1$ and $\tilde{\alpha}_2 = v_2 x_2$, where v_i is a constant coefficient ($i = 1, 2$). On this subject, see Bischi and Lamantia (2002, pp. 200–201).

Remark 3 Excepting the Cournot-Nash equilibrium point, there exist three steady-state points in this system: $(0, 0)$, $((a - c_1)/(2b), 0)$, and $(0, (a - c_2)/(2b))$. In each point at least one of the outputs is zero. This is hardly consistent with the realistic situation, so we restrict our attention to the Cournot-Nash equilibrium hereafter.

3 N-firm Cournot oligopoly and global stability

In this section we deal with an n-firm oligopoly model. The asymptotic behavior of the stationary point is examined via the method of the Lyapunov function.

The market price is given by

$$p = a - b(x_1 + x_2 + \cdots + x_n), \ a > 0, b > 0, \tag{24}$$

and we assume that the cost function of firm i is linear,

$$C_i(x_i) = c_i x_i, \ c_i > 0. \tag{25}$$

Using these functions, we obtain the profit function of firm i,

$$\pi_i = [a - b(x_1 + x_2 + \cdots + x_n)]x_i - c_i x_i \tag{26}$$

As mentioned in the previous section, we assume the Cournot conjecture: any firm believes that all its rivals' output levels remain constant in response to a change in its own output level. Moreover, we assume that any firm is rational in the sense that it maximizes its own profit. Thus we can obtain the reaction function of firm i,

$$x_i = R_i\left(\sum_{j\neq i} x_j\right) = \frac{a - c_i}{2b} - \frac{1}{2}\sum_{j\neq i} x_j \tag{27}$$

The system of equations

$$x_i = \frac{a - c_i}{2b} - \frac{1}{2}\sum_{j\neq i} x_j, \quad (i = 1, \cdots, n), \tag{28}$$

defines the Cournot-Nash equilibrium. The above system can be rewritten as

$$\frac{1}{2}x_i + \frac{1}{2}\sum_{j=1}^{n} x_j = \frac{a - c_i}{2b}, \quad (i = 1, \cdots, n), \tag{29}$$

which leads to

$$\sum_{j=1}^{n} x_j = \frac{1}{n+1}\frac{na - \sum_{j=1}^{n} c_j}{b}. \tag{30}$$

Thus we can obtain the unique Cournot-Nash equilibrium point $(x_1^*, x_2^*, \cdots, x_n^*)$, where

$$x_i^* = \frac{a - c_i}{2b} - \frac{1}{n+1}\frac{na - \sum_{j=1}^{n} c_j}{b}, \quad (i = 1, \cdots, n). \tag{31}$$

Let us now turn to the study of dynamic properties of the Cournot model. We assume the same type of adjustment process as in the preceding section:

$$\dot{x}_i = \alpha_i \left[R_i \left(\sum_{j \neq i} x_j \right) - x_i \right] x_i, \tag{32}$$

where α_i is a positive constant and called the speed of adjustment of firm i. This equation means that each firm adjusts the actual level of output to the desired level of output.

Note that the stationary state of the above system is given by $x_i = R_i(\sum_{j \neq i} x_j)$, $i = 1, \cdots, n$. This fact implies that the stationary state exactly corresponds to the Cournot-Nash equilibrium.

Applying Lyapunov's second method, we can obtain the following:

Proposition 3 *The Cournot-Nash equilibrium is globally asymptotically stable in R^n_{++}.*

Proof To apply Lyapunov's second method we have to check the two conditions mentioned in Section 2. The function

$$\begin{aligned} V(x_1,x_2,\cdots,x_n) = {} & (x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 + \cdots + (x_n - x_n^*)^2 \\ & + (x_1 - x_1^*)(x_2 - x_2^*) + (x_1 - x_1^*)(x_3 - x_3^*) + \cdots + (x_1 - x_1^*)(x_n - x_n^*) \\ & + (x_2 - x_2^*)(x_3 - x_3^*) + (x_2 - x_2^*)(x_4 - x_4^*) + \cdots + (x_2 - x_2^*)(x_n - x_n^*) \\ & + (x_3 - x_3^*)(x_4 - x_4^*) + (x_3 - x_3^*)(x_5 - x_5^*) + \cdots + (x_3 - x_3^*)(x_n - x_n^*) \\ & + \cdots + (x_{n-1} - x_{n-1}^*)(x_n - x_n^*). \end{aligned} \tag{33}$$

is a candidate Lyapunov function. Since it is a quadratic form, we can rewrite in the matrix form as follows:

$$V(x_1, x_2, \cdots, x_n) = z^T A z, \tag{34}$$

where the superscript T is a transpose operator,

$$\hat{z}^T = (x_1 - x_1^*, x_2 - x_2^*, \cdots, x_n - x_n^*),$$

$$A = \begin{pmatrix} 1 & 1/2 & 1/2 & \dots & 1/2 \\ 1/2 & 1 & 1/2 & \dots & 1/2 \\ 1/2 & 1/2 & 1 & \dots & 1/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & 1/2 & \dots & 1 \end{pmatrix}.$$

First, we shall verify that $V(x_1^*, x_2^*, \cdots, x_n^*) = 0$ and $V(x_1, x_2, \cdots, x_n) > 0$ for all $(x_1, x_2, \cdots, x_n) \neq (x_1^*, x_2^*, \cdots, x_n^*)$. From the definition, it is clear that $V(x_1^*, x_2^*, \cdots, x_n^*) = 0$. Let $|A_k|$ be a leading principal minor of order k for $k = 1, 2, \cdots, n$. That is,

$$|A_k| = \begin{vmatrix} 1 & 1/2 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & 1/2 & \cdots & 1 \end{vmatrix} = \left(\frac{1}{2}\right)^k \left.\begin{vmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{vmatrix}\right\} k \tag{35}$$

To the first row we add all the rows except the first and we have

$$|A_k| = \left(\frac{1}{2}\right)^k (k+1) \left.\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{vmatrix}\right\} k \tag{36}$$

Then we subtract the first row from the all the rows except the first and obtain an upper triangular matrix. With some rearrangement, we arrive at the conclusion that $|A_k| = (1/2)^k(k+1) > 0$ for $k = 1, 2, \cdots, n$. This means that V is positive definite, i.e., $V(x_1, \cdots, x_n) > 0$ for all $(x_1, \cdots, x_n) \neq (x_1^*, \cdots, x_n^*)$. [3]

To check the second condition for Lyapunov's second theorem, we take the derivative

[3] A necessary and sufficient condition for a symmetric matrix A to be positive definite is that the determinant of every leading principal submatrix be positive. On this point, see Hoy, Livernois, McKenna, Rees, and Stengos (2001, Chapter 10.3), for example.

of V with respect to time:

$$
\begin{aligned}
\dot{V}(x_1, x_2, \cdots, x_n) = {} & 2(x_1 - x_1^*)\dot{x}_1 + 2(x_2 - x_2^*)\dot{x}_2 + \cdots + 2(x_n - x_n^*)\dot{x}_n \\
& + \dot{x}_1(x_2 - x_2^*) + \dot{x}_1(x_3 - x_3^*) + \cdots + \dot{x}_1(x_n - x_n^*) \\
& + (x_1 - x_1^*)\dot{x}_2 + (x_1 - x_1^*)\dot{x}_3 + \cdots + (x_1 - x_1^*)\dot{x}_n \\
& + \dot{x}_2(x_3 - x_3^*) + \dot{x}_2(x_4 - x_4^*) + \cdots + \dot{x}_2(x_n - x_n^*) \\
& + (x_2 - x_2^*)\dot{x}_3 + (x_2 - x_2^*)\dot{x}_4 + \cdots + (x_2 - x_2^*)\dot{x}_n \\
& + \dot{x}_3(x_4 - x_4^*) + \dot{x}_3(x_5 - x_5^*) + \cdots + \dot{x}_3(x_n - x_n^*) \\
& + (x_3 - x_3^*)\dot{x}_4 + (x_3 - x_3^*)\dot{x}_5 + \cdots + (x_3 - x_3^*)\dot{x}_n \\
& + \cdots + \dot{x}_{n-1}(x_n - x_n^*) + (x_{n-1} - x_{n-1}^*)\dot{x}_n. \\
= {} & \dot{x}_1[2(x_1 - x_1^*) + (x_2 - x_2^*) + \cdots + (x_n - x_n^*)] \\
& + \dot{x}_2[(x_1 - x_1^*) + 2(x_2 - x_2^*) + \cdots + (x_n - x_n^*)] \\
& + \cdots + \dot{x}_n[(x_1 - x_1^*) + (x_2 - x_2^*) + \cdots + 2(x_n - x_n^*)] \qquad (37)
\end{aligned}
$$

Note that the adjustment process governed by (32) can be transformed into

$$
\dot{x}_i = -\frac{1}{2}\alpha_i \left[\sum_{j \neq i} (x_j - x_j^*) + 2(x_i - x_i^*) \right] x_i, \qquad i = 1, 2, \cdots, n. \tag{38}
$$

Substituting (38) into (**??**) gives

$$
\dot{V}(x_1, x_2, \cdots, x_n) = -\frac{2(\dot{x}_1)^2}{\alpha_1} - \frac{2(\dot{x}_2)^2}{\alpha_2} - \cdots - \frac{2(\dot{x}_n)^2}{\alpha_n} < 0 \quad \text{for all } x \in R_{++}^n - \{x^*\} \tag{39}
$$

From the above arguments, we can say that V is a Lyapunov function of the adjustment process. This completes our proof.

4 A modified model with time lags

In this section we shall develop the previous analysis by introducing discrete time lags, which are induced by the inherent properties of production activities. There are several reasons for the existence of time lags. First of all, firms must collect an enormous amount of data to analyze the state of the market. Such an information collecting process needs time. Second, the decision process is also time-consuming; for example, alternative plans are considered extensively and thoroughly, and the consequences are evaluated carefully and completely. Finally, firms are engaged in the implementation process. They must flawlessly execute the designed plans. Since in reality they are hierarchical organizations, which have many different divisions, departments, and sections, they devote their time to the task of coordinating every section and activating the organization as a whole. These facts suggest that the time lags in the adjustment process are unavoidable in the real world.

We shall consider the following system of delay differential equations:

$$\begin{aligned}
\dot{x}_1(t) &= \alpha_1(R_1(x_2(t-l_1)) - x_1(t-\tau_1)), \\
&= \alpha_1\left(-x_1(t-\tau_1) - \frac{1}{2}x_2(t-l_1) + \frac{a-c_1}{2b}\right)x_1(t) \qquad (40a) \\
\dot{x}_2(t) &= \alpha_2(R_2(x_1(t-l_2)) - x_2(t-\tau_2)) \\
&= \alpha_2\left(-\frac{1}{2}x_1(t-l_2) - x_2(t-\tau_2) + \frac{a-c_2}{2b}\right)x_2(t) \qquad (40b)
\end{aligned}$$

where τ_i is a time lag that firm i needs in information processing regarding its own situation, and l_i is a time lag that firm i suffers in information processing concerning the movements of the rival firm.

This type of model was investigated by Howroyd and Russell (1984) and Russell, Rickard, and Howroyd (1986). While our model shares many common elements with their models, we have to emphasize a key difference that we believe to be of crucial importance. That is, unlike their models, our model includes a nonlinear relationship, which is due to the assumption of the growth-rate rule regarding the adjustment process. This feature, therefore, requires us to apply nonlinear dynamical systems theory to our model. In particular, we concentrate our attention on whether our system can generate the endogenous and perpetual fluctuations in the duopoly market.

Let us now start our analysis of system (40). The model to consider here is precisely identical to a model developed by Shibata and Saito (1980).[4] Their model is an interesting contribution to the field of mathematical biology. They investigated the population dynamics of two competing species with fixed time lags and showed the appearance of limit cycles and chaotic attractors by means of numerical simulations.

Following Shibata and Saito, we hereafter resort to numerical experiments to examine the dynamic properties of our system. This is because the qualitative analysis of delay differential equations is quite difficult. We set the following parameter values:

$$a = 3, \;\; b = 1, \;\; c_1 = c_2 = 1,$$

$$\alpha_1 = 2, \;\; \alpha_2 = 2, \;\; \tau_2 = 0.88, \;\; l_1 = 0.31, \;\; l_2 = 0.29.$$

Moreover, the following set of initial conditions are assumed:

$$x_1(s) = 0.8 \;\; \text{for} \; -l_1 \leq s \leq 0, \;\; x_2(s) = 0.8 \;\; \text{for} \; -l_2 \leq s \leq 0.$$

Figures 3–6 illustrate the emergence of a period-doubling route to chaos in our system. In these figures we show the projections of the solutions of our system on the $(\ln x_1, \ln x_2)$-plane and the power spectra of $x_1(t)$ for different values of τ_1.

Figure 3 is obtained by fixing $\tau_1 = 0.9$. Note that we plot only the final trajectory by excluding the transient behavior. Figure 3A displays a limit cycle, whose phase flow moves in a clockwise direction. The motion is periodic and persistent. In Figure 3B show a basic frequency, which corresponds to the highest peak in the power spectrum at $f_0 = 0.0016$.

Increasing the value τ_1 further to 0.95, we can obtain Figure 4. Here we can see that the limit cycle observed in Figure 3 loses its stability and a period-2 cycle with two peaks and two troughs per cycle appears around it. This situation is confirmed in

[4] Strictly speaking, there is a difference between our model and theirs. While we consider four time lags in our Cournot adjustment model, they introduce two time lags in their population dynamics model. In terms of our model, they assume that $l_1 = l_2 = 0$.

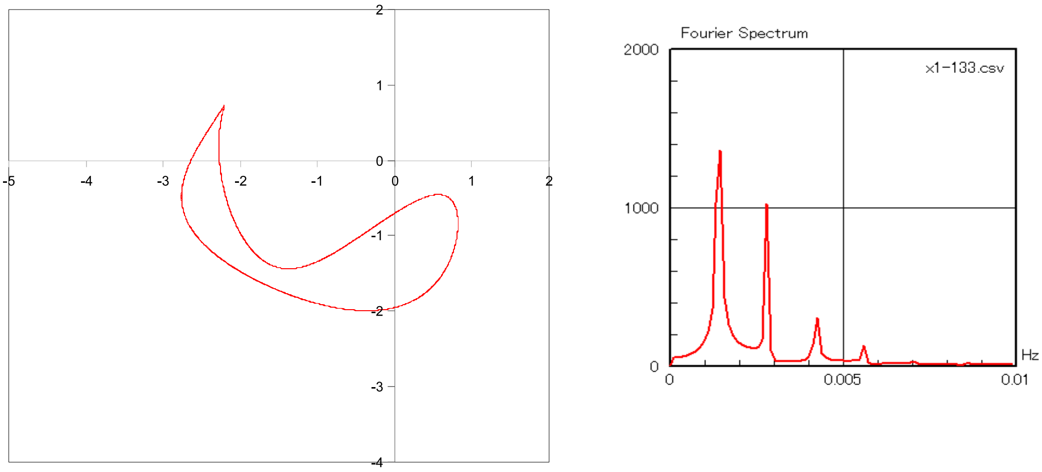

Figure 3: Limit cycle: $\tau_1 = 1.33$

Figures 4A.[5] Furthermore, Figure 4B shows that the bifurcation doubles the number of sharp frequency components.

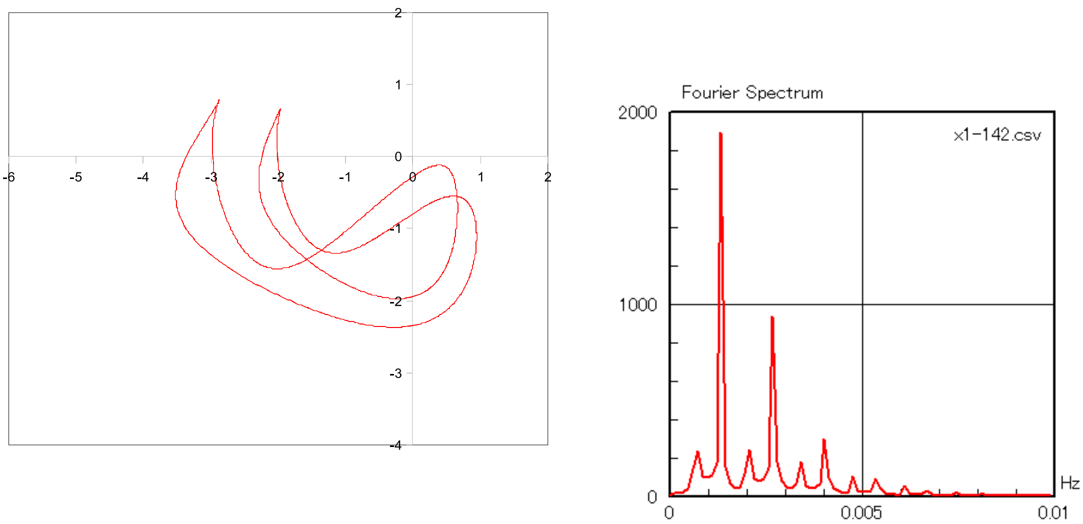

Figure 4: Period-2 cycle: $\tau_1 = 1.25$

When $\tau_1 = 0.95$ is increased to 0.975, the system exhibits a period-4 cycle as

[5] At first sight the trajectory seems to cross itself again and again. The self-intersection is, however, an artifact of the projection onto the $(\ln x_1, \ln x_2)$ plane. In the full dimensional state space no self-intersections occur. This property applies to the below figures.

seen in Figure 5. In Figure 5A we depict the projection of the period-4 cycle onto the $(\ln x_1, \ln x_2)$ plane. The period-4 cycle has four peaks and four troughs within a cycle. Figure 5B shows that the bifurcation doubles the number of sharp frequency components again.

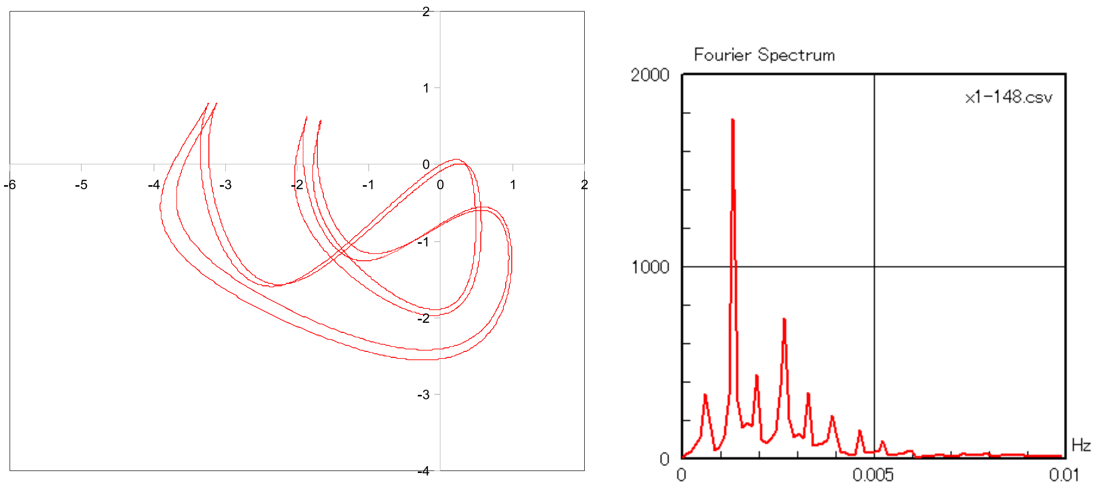

Figure 5: Period-4 cycle: $\tau_1 = 1.3$

A more interesting state arises when τ_1 is increased from 0.975 to 1.1: Figure 6 shows a strange attractor in our system. In Figure 6A we observe complex behavior at first glance. In Figure 6B displays the coexistence of sharp peaks and the broad background, which is a noteworthy feature in our numerical results. This type of strange attractors is named 'phase coherence' by Farmer, Crutchfield, Froehling, Packard, and Shaw (1980). As demonstrated in their paper, both the Rössler attractor and the Lorenz attractor are famous examples of phase coherence. In order to convince the exact evidence for deterministic chaos, we examine the Lyapunov characteristic exponents, which measure the rates of divergence or convergence of orbits starting from nearby initial conditions. Any system containing at least one positive Lyapunov characteristic exponent is said to be chaotic since it has sensitive dependence on initial conditions. In fact, we estimate that the largest Lyapunov exponent is 1.42 when $\tau_1 = 1.1$.[6] This result therefore suggests the presence of chaos in our numerical example.

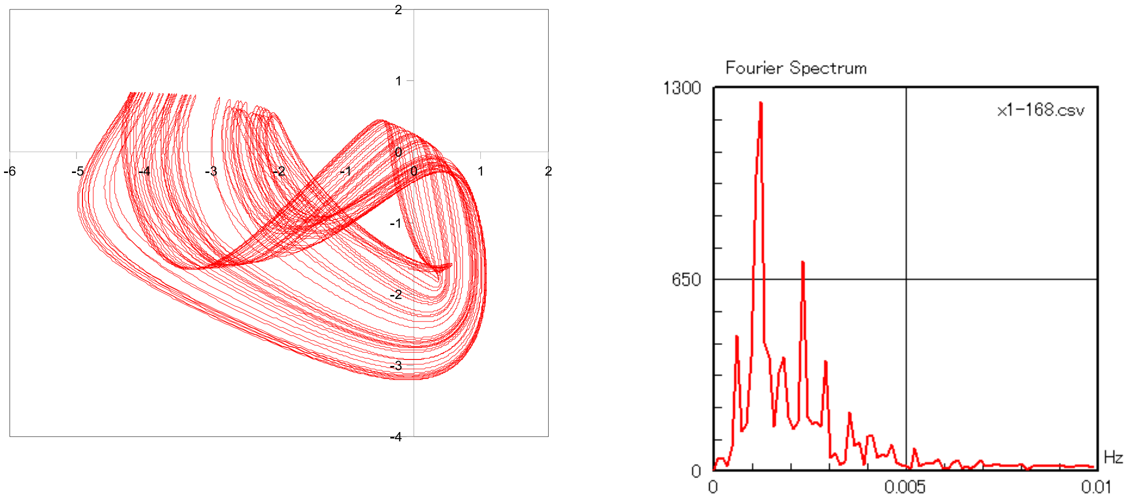

Figure 6: Chaotic attractor: $\tau_1 = 1.6$

Let us now attempt to extend our numerical examination into the adjustment speed; we choose α_1 as a new bifurcation parameter and fix the parameter values as:

$$a = 3, \quad b = 1, \quad c_1 = c_2 = 1,$$

$$\alpha_2 = 2.0, \quad \tau_1 = 1.58, \quad \tau_2 = 0.88, \quad l_1 = 0.31, \quad l_2 = 0.29.$$

Again we can observe a period doubling route to chaos in the case of α_1. With Figure 7 we provide a bifurcation diagram with respect to α_1. We compute the time series of $x_1(t)$ with a set of initial conditions for a given value α_1, and then plot the local maximum and minimum values of the time series of x_1 as a function of the parameter α_1 being varied. This figure shows a clear impression of the period doubling bifurcation cascade when α_1 is increased from 1.3 to 2.0.

[6] In this case, we used the Sano-Sawada method to compute the largest Lyapunov exponent. The method is able to determine the spectrum of several Lyapunov exponents from the observed time series of a single variable. On this point, see Sano and Sawada (1985).

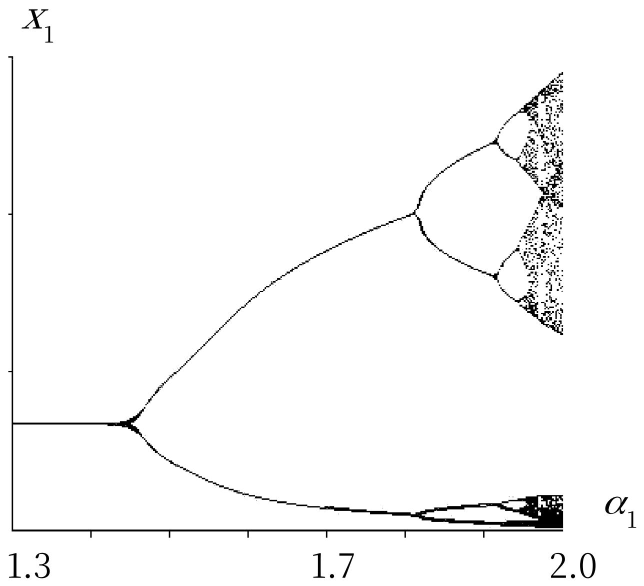

Figure 7: Bifurcation diagram with respect to $\alpha_1 \in [1.3, 2.0]$

From the above numerical simulations, we have two important observations about the dynamic adjustment process. First, an increase in the time lag τ_1 could be a destabilizing factor; If firm 1 requires a long delay in information processing, the Cournot-Nash equilibrium loses its stability and yields complex dynamics. Second, an increase in the adjustment speed α_1 also could destabilize the market dynamic behavior. This result implies that a rapid response of firm 1 leads to chaotic fluctuations in the market. The reason is explained intuitively as follows. Notice that firm 1 suffers time delays in collecting, evaluating, and sharing the information. If the time lags of information are sufficiently long, the firm tends to make improper decisions by using the old and unsuitable information. In this circumstance, it is no wonder that the adjustment process in the duopoly market could generate persistent oscillations and even complex dynamics.

5 Conclusion

We have investigated the dynamic nature of the adjustment process by adopting a Cournot model with linear demand and constant marginal costs. In particular, we considered two types of models. First, we formulated a dynamic adjustment process without time lags. In this case we established the local and the global stability of the Cournot-Nash equilibrium. Second, we examined a dynamic adjustment process with four time lags. In this process we observed that irregular and chaotic fluctuations could emerge in the duopoly market by applying the results of Shibata and Saito (1980). Moreover, in the second type model we detected the period-doubling routes to chaos with respect to the time delay and the adjustment speed.

The comparison between the results of the two models shows that the introduction of time lags in the adjustment process changes the dynamic properties of the Cournot-Nash equilibrium substantially. In particular, our numerical simulations indicate that the existence of time lags could be a destabilizing factor. From the practical point of view, there is no doubt that time lags are inherent and unavoidable in the adjustment process. Consequently, this fact leads us to the conclusion that economists must pay much more attention to the lag structure in analyzing the dynamic behavior of the economy. We hope that this paper will inspire further research efforts in microeconomic dynamics.

Acknowledgement

This research was supported by the International Visiting Scholar Program of the College of Economics, Nihon University. Orlando visited Nihon University in April 2023, while Yoshida visited the University of Bari in August 2023, both made possible by research funds from Nihon University.

References

1. Bischi, G.I. and Kopel, M. (2001) Equilibrium selection in a nonlinear duopoly game with adaptive expectations *Journal of Economic Behavior and Organization*, 46, pp.73–100

2. Bischi, G.-I. and Lamantia, F. (2002), Nonlinear duopoly games with positive cost externalities due to spillover effects, *Chaos, Solitons & Fractals*, 13(4), pp. 701-721

3. Chiarella, C., and Khomin, A. (1996). An analysis of the complex dynamic behaviour of nonlinear oligopoly models with time delays. *Chaos, Solitons & Fractals*, 7(12), 2049-2065.

4. Cournot, A. A. (1838), *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, Paris, Hachette. English edition translated by Nathaniel T. Bacon (1960), *Researches into the Mathematical Principles of the Theory of Wealth*, New York: Kelly.

5. Farmer, D., Crutchfield, J., Froehling, H., Packard, N., and Shaw, R. (1980). Power spectra and mixing properties of strange attractors. *Annals of the New York Academy of Sciences*, 357(1), 453-471.

6. Fisher, F. M. (1961). The stability of the Cournot oligopoly solution: The effects of speeds of adjustment and increasing marginal costs. *The Review of Economic Studies*, 28(2), 125-135.

7. Furth, D. (2009). Anything goes with heterogeneous, but not always with homogeneous oligopoly. *Journal of Economic Dynamics and Control*, 33(1), 183-203.

8. Gallet, C. A. and List, J. A. (2001) Market share instability: an application of

unit root tests for the cigarette industry, *Journal of Economics and Business*¥, 53, 473–80.

9. Guckenheimer and Holmes (1997) *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag.
10. Hahn, F. H. (1962). The stability of the Cournot oligopoly solution. *The Review of Economic Studies*, 29(4), pp. 329-331.
11. Huang, W. (2002) Controlling Chaos Through Growth Rate Adjustment Discrete dynamics in Nature and Society, 7 (3) 191-199
12. Hoy, M., Livernois, J., McKenna, C., Rees, R., and Stengos, A. (2001), *Mathematics for Economics.* Second Edition, MIT Press.
13. Howroyd, T.D. and Russell, A.M. (1984), Cournot oligopoly models with time delays, *Journal of Mathematical Economics*, 13 (2) 97–103
14. Kopel, M. (1996) Simple and complex adjustment dynamics in Cournot duopoly models *Chaos, Solitons & Fractals*, 7 (2), pp. 2031-2048
15. Lorenz, E.N. (1963). Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, 20, pp.130–141.
16. MacDonald, N. (1978), *Time Lags in Biological Models*, Lecture Notes in Biomathematics No 27, Springer-Verlag.
17. Mazzucato, M. and Semmler, W. (1999) Market share instability and stock price volatility during the industry life-cycle: the US automobile industry, *Journal of Evolutionary Economics*, 9, 67–96.
18. Nash, J.F. (1951) Non-cooperative games. *Annals of Mathematics*, 54 (2), pp. 286-295.

19. Okuguchi, K. (1976). *Expectations and stability in oligopoly models* (Vol. 138), Springer.

20. Perko, L. (1996). *Differential Equations and Dynamical Systems.* Springer-Verlag.

21. Puu, T. (1991) Chaos in duopoly pricing, *Chaos, Solitons & Fractals*, 1, pp. 573–581

22. Puu, T. and Sushko, I. (eds.)(2002), *Oligopoly Dynamics: Models and Tools*, Springer-Verlag.

23. Rand, D. (1978). Exotic phenomena in games and duopoly models. *Journal of Mathematical Economics*, 5(2), 173-184.

24. Rössler, O. E. (1976). An equation for continuous chaos, *Physics Letters A* 57 (5), pp. 397–398.

25. Russell, A.M., Rickard, J.A., and Howroyd, T.D. (1986), The effects of delays on the stability and rate of convergence to equilibrium of oligopolies, *Economic Record*, 62 (177), pp.194–198

26. Sano, M and Sawada, Y. (1985), Measurement of the Lyapunov spectrum from a chaotic time series, Physical Review Letters, 55 (10) pp.1082-1085

27. Shibata, A. and Saito, N. (1980), Time delays and chaos in two competing species, *Mathematical Biosciences*, 51, pp.199–211

Research Institute of Economic Science
College of Economics, Nihon University

1-3-2 Kandamisaki-cho, Chiyoda-ku, Toyko 101-8360 JAPAN
Phone: 03-3219-3309　Fax: 03-3219-3329
E-mail: keikaken.eco@nihon-u.ac.jp
http://www.eco.nihon-u.ac.jp/research/economic/