Simulation Techniques and Systems Analysis in Economic-Demographic Planning

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C O N T E N T S

T A B L E S

F I G U R E

A B S T R A C T ──────────────

In the present paper, some of the simulation techniques commonly used are critically reviewed, and their advantages and disadvantages are briefly discussed. As regards model estimation and validation, both recursive and non-recursive systems are evaluated from the standpoint of population planning, drawing upon a few illustrative models fitted to the Japanese annual time-series data over the period 1965-1982. In the last section of this paper, both Jacobi and Gauss-Seidel iterative methods for solving a simultaneous equations model are compared to examine their usefulness and limitations.

I. Introduction

Since the early 1970s, development planners in developing countries have been increasingly aware of the serious challenge which rapid population growth poses to their social and economic developmental efforts. In recent years, they have also started to pay greater recognition to the importance of feedbacks from socio-economic variables to population changes. Despite such increased awareness and recognition among the government officials concerned with development planning, demographic factors have been integrated into actual development planning schemes to only a limited extent. One of the major obstacles to such integration exercises lies in the fact that it is difficult for the planners to form and utilize a mental model which would accurately reflect a host of complex feedback loops among population and socio-economic variables. To cope with this problem, therefore, formal mathematical models are required to facilitate them to gain important insights into demographic-economic dynamics that may not be intuitively obvious or derivable using analytical methods.

Models, which might be considered as a miniature version of a complex economy, bring together the main relationships among population and socio-economic variables expressed in a quantitative form. Such models could be used to show the future course of events resulting from the alternative choice of policy variables. For instance, the models could assist the planner by demonstrating that the economic and social policy goals established by the government would be harder to achieve under certain demographic conditions. Conversely, they could also assist the planner by indicating the degree to which those goals could be achieved more rapidly by making greater demographic investments (ESCAP, 1977).

In view of such potential usefulness, a number of demographic-economic models have been built, and applied to various developing countries in recent years (ESCAP, 1983). It should be noted, however, that given the long time-period involved in economic-demographic interaction and the obvious uncertainties inherent in looking very far into the future, most of these models are of a simulation rather than forecasting nature for policy analysis.

In the present paper, we will review some of the commonly-used simulation techniques, and identify both advantages and disadvantages

of each technique. In the next section, both recursive and non-recursive systems will be briefly compared in terms of model estimation and validation by drawing upon a few illustrative cases. Section III will examine the usefulness and limitations of two iterative methods for solving a simultaneous equations model.

II. Comparison of Recursive and Non-recursive Systems

Although model parameters may be estimated by assuming certain reasonable values or using values taken from other studies, the parameters of most currently available population-development models have been estimated on the basis of standard econometric techniques (UNFPA, 1980). In general, parameters in a simultaneous equations system should not be estimated by ordinary least squares (OLS) if the econometrician desires consistent estimators. However, in a special type of simultaneous equations model known as a recursive model, the use of OLS is appropriate.

A recursive model is one in which there may be a simultaneous dependency of some endogenous variables on others, but in which this cause and effect role is only in one direction. The causal chain within the model must be from pre-determined variables to the first endogenous variable, then from this first endogenous variable and perhaps, other pre-determined variables to the second endogenous variable, and so forth, to the last endogenous variable. There must not be any feedback from any endogenous variable to one of a lower order in the causal chain.

The special characteristics of a recursive model can be seen by a simple example. Consider a compact interdependent simultaneous equations model as follows:

$$(1) \quad LM_t = a + b\ PM_t \qquad : \text{Male labor force}$$

$$(2) \quad LF_t = a + b\ PF_t \qquad : \text{Female labor force}$$

$$(3) \quad L_t = LM_t + LF_t \qquad : \text{Total labor force}$$

$$(4) \quad \ln Y_t = a + b\ \ln K_t + c\ \ln L_t : \text{Production function}$$

$$(5) \quad \ln[(0.4 - SR_t)/SR_t] =$$
$$a + b\ (Y_t/EAC_t) + c\ DM_t \quad : \text{Saving rate function}$$

$$(6) \quad S_t = SR_t \cdot Y_t \qquad\qquad : \quad \text{Gross saving}$$

$$(7) \quad K_t = K_{t-1} + I_t - D_t \qquad : \quad \text{Total capital stock}$$

$$(8) \quad D_t = a + b \, K_{t-1} \qquad\quad : \quad \text{Depreciation function}$$

$$(9) \quad I_t = S_t \qquad\qquad\qquad\quad : \quad \text{Equilibrium condition}$$

where LM = male labor force in thousand persons, LF = female labor force in thousand persons, PM = male population aged 15-64 in thousand persons, PF = female population aged 15-64 in thousand persons, L = total labor force in thousand persons, Y = real GNP in billions of 1975 yen, K = total capital stock in billions of 1975 yen, SR = saving rate, EAC = total population adjusted by equivalent adult consumer units (0.25 for those aged 0-4, 0.4 for 5-9, 0.6 for 10-14), DM = dummy variable representing the oil crisis upon an economy, D = depreciation in billions of 1975 yen, I = gross investment in billions of 1975 yen, S = gross savings in billions of 1975 yen, and t = time.

Ordinary least squares gives biased estimates when applied to such a model because the explanatory endogenous variables are not independent of the error terms. For instance, the error term for Eq. (4) affects Y, and Y affects K through the saving rate equation. Thus, a linkage is developed between the explanatory variable and the disturbance in the same production equation.

Suppose, however, that current production depends on the level of capital stock in the previous period and labor force in the current period. Then, the model is changed so that the endogenous variable, K_t, in the production function is replaced by a pre-determined variable, K_{t-1}. The system is no longer interdependent but recursive.

A wide variety of estimation techniques are now available to deal with the problem of biased results to be yielded by OLS in the case of simultaneous non-recursive systems. These estimation techniques include the method of two-stage least squares (2SLS), the method of indirect least squares (ILS), and the limited information maximum likelihood (LIML) method. It is known that in the case of exact identification of structural equations, the estimates of the structural form coefficients are identical by 2SLS, ILS, or LIML (Goldberger, 1964). However, when the structural equations are over-identified and the value of the variance ratio is greater than 1, then

ILS is not applicable and estimates obtained by LIML and 2SLS will be different. Since 2SLS estimators are easier to understand and to calculate, they are more commonly used in the over-identified case than the LIML estimators (Murphy, 1973).

For illustrative purposes, we have used OLS to estimate the structural equations of the recursive model, by drawing upon the Japanese annual time-series data over the period 1965-1982. As regards the non-recursive version of the present model, however, the main structural equations such as the production function are over-identified, so that we have applied 2SLS to the same data set to estimate the coefficients.

Table 1 presents the estimated behavioral equations for the recursive version of the model in question. Standard errors are noted by the figures in parentheses under the estimated regression co-efficients. All the estimated coefficients conform to a priori expectations, although the proportion of the total variation that has been explained is rather low for both saving rate and female labor force functions, as compared with the remaining estimated functions.

For the non-recursive version of the present model, we have used 2SLS to estimate the coefficients for both production and saving rate equations. The estimated results for these equations are as below:

Production function:

$$\ln Y_t = 0.189 + 0.451 \ln K_t + 0.549 \ln L_t$$

$$\overline{R}^2 = 0.993$$

Saving rate function:

$$\ln[(0.4 - SR_t)/SR_t] = 0.864 - 2.365 (Y_t/EAC_t)$$

$$+ 1.847 DM_t$$

$$\overline{R}^2 = 0.603$$

As was the case for the recursive type, the estimated coeffi-cients in the above two equations are completely in accord with

theoretical predictions. The size of each estimated coefficient, however, differs with that for the recursive model. As regards the Cobb-Douglas production function, for instance, the coefficient for capital stock is approximately 10 percent larger for the nonrecursive case than for the recursive case. Note that no t-values for the significance of the estimates of the coefficients are applicable

Table 1. Estimated OLS Results for the Recursive Model

(i)　Male labor force function:

$$LM_t = -709.73 + 0.912\ PM_t$$
$$\quad\ \ (965)\quad\ (0.026)$$

$$D - W = 0.565\ ;\ \bar{R}^2 = 0.986$$

(ii)　Female labor force function:

$$LF_t = 4437 + 0.426\ PF_t$$
$$\quad\ (2579)\ (0.068)$$

$$D - W = 0.400\ ;\ \bar{R}^2 = 0.690$$

(iii) Production function:

$$\ln Y_t = 0.313 + 0.408\ \ln K_{t-1} + 0.592\ \ln L_t$$
$$\quad\ \ (0.02)\ \ (0.01)\qquad\quad\ (0.02)$$

$$D - W = 0.710\ ;\ \bar{R}^2 = 0.989$$

(iv)　Saving rate function:

$$\ln[(0.4 - SR_t)/SR_t] = 0.449 - 1.982(Y_t/EAC_t) + 1.515\ DM_t$$
$$\qquad\qquad\qquad\ \ (0.49)\ \ (0.39)\qquad\qquad\ (0.27)$$

$$D - W = 1.557\ ;\ \bar{R}^2 = 0.639$$

(v)　Depreciation function:

$$D_t = -572.7 + 0.064\ K_{t-1}$$
$$\quad\ \ (667)\quad\ (0.002)$$

$$D - W = 0.207\ ;\ \bar{R}^2 = 0.983$$

to 2SLS results because the distribution of structural form estimators computed by 2SLS on the basis of a small finite sample such as this is not normal. For this reason, no standard error is shown in the above two equations.

As mentioned earlier, OLS yields biased results in the case of simultaneous non-recursive systems. Despite such statistical problems, many large-scale econometric demographic models have been estimated by OLS mainly because it is the simplest estimation technique. In large models estimated by 2SLS, the exclusion or addition of one exogenous variable from a certain equation often requires the reestimation of all the equations containing endogenous variables as explanatory variables. For comparative purposes, we have estimated the production function for the non-recursive model on the basis of OLS. The estimated result is:

Production function:

$$\ln Y_t = 0.189 + 0.452 \ln K_t + 0.548 \ln L_t$$
$$ (0.016) \ (0.01) (0.02)$$

$$D - W = 0.706 \ ; \ \overline{R}^2 = 0.993$$

One can easily note that the estimated coefficients of this equation are virtually the same with those for OLS.

Up to this point, we have discussed various technical problems related to the estimation of model coefficients. Let us now consider some of the difficulties commonly observed in connection with model validation. To facilitate the discussions which will follow, we have undertaken a series of final tests of the three different types of the model, namely, the recursive type, the non-recursive type based on 2SLS, and the non-recursive type estimated by OLS. Results of the final tests show that these three versions are equipped with a comparable level of simulatability. To briefly illustrate this conclusion, we will examine the performance of selected key variables.

Table 2 shows actual values of the variables used for estimating the structural form coefficients. Table 3 compares the percent deviations of simulated values from observed ones for selected variables in the three different cases, i.e., the recursive type, the non-recursive type by 2SLS, and the non-recursive type by OLS.

Table 2. Original Data Matrix Used for Estimation, 1965-1982

Year	LM	LF	L	Y	D	SR	I	K
1965	28,840	19,030	47,870	68,992	4,471	0.282	19,458	70,950
1966	29,420	19,490	48,910	76,325	5,175	0.290	22,163	87,938
1967	29,920	19,910	49,830	84,567	6,076	0.320	27,038	108,899
1968	30,580	20,030	50,610	95,319	7,157	0.342	32,570	134,313
1969	30,910	20,070	50,980	107,035	8,378	0.355	37,954	163,889
1970	31,290	20,240	51,530	117,591	9,848	0.379	44,576	198,617
1971	31,820	20,050	51,870	123,104	11,063	0.360	44,350	231,904
1972	32,170	19,820	51,990	134,147	12,997	0.364	48,843	267,750
1973	32,790	20,470	53,260	145,977	15,495	0.383	55,884	308,139
1974	33,110	19,990	53,100	144,166	18,006	0.361	52,102	342,235
1975	33,360	19,870	53,230	147,655	19,313	0.329	48,536	371,457
1976	33,680	20,100	53,780	155,502	21,288	0.325	50,513	400,682
1977	33,810	20,700	54,510	163,752	24,033	0.324	53,011	429,660
1978	34,060	21,250	55,310	172,133	26,379	0.335	57,714	460,994
1979	34,370	21,600	55,970	181,137	28,939	0.343	62,132	494,187
1980	34,650	21,850	56,500	189,787	31,641	0.330	62,614	525,160
1981	34,980	22,090	57,070	197,462	34,799	0.328	64,696	555,057
1982	35,220	22,520	57,740	203,908	37,640	0.322	65,748	583,165

Table 3. Performance of Three Different Types of the Model

(Unit: %)

Year	LM	LF	L	Y	D	SR	I	K
				Recursive type				
1965	-0.0	-0.0	-0.0	-0.0	-0.0	0.0	0.0	0.0
1966	-0.1	-1.1	-0.5	0.5	4.7	1.5	2.1	0.2
1967	-0.2	-2.1	-0.9	-0.0	7.0	-4.3	-4.3	-1.3
1968	-1.0	-1.7	-1.2	-3.2	7.9	-7.7	-11.2	-4.1
1969	-0.7	-1.0	-0.8	-6.8	8.4	-8.6	-16.0	-7.4
1970	-1.2	-1.1	-1.1	-9.1	7.6	-13.2	-23.5	-11.5
1971	-1.8	0.7	-0.8	-6.6	9.9	-5.3	-12.2	-12.9
1972	-1.3	3.1	0.4	-8.7	7.3	-4.6	-13.7	-14.3
1973	-2.3	0.6	-1.2	-11.5	2.4	-8.3	-20.8	-16.4
1974	-2.3	3.5	-0.0	-4.3	-1.1	-0.8	-5.2	-15.5
1975	-1.0	4.8	1.2	-1.1	2.7	-21.1	-22.4	-17.5
1976	-1.1	4.2	0.9	-3.2	-0.8	-17.3	-21.1	-19.0
1977	-0.7	1.8	0.3	-5.5	-7.2	-14.9	-21.2	-20.0
1978	-0.5	-0.3	-0.4	-7.7	-10.9	-17.0	-25.9	-21.3
1979	-0.5	-1.4	-0.8	-10.1	-14.8	-17.7	-29.6	-22.7
1980	-0.3	-1.9	-0.9	-12.1	-18.7	-11.5	-24.9	-23.2
1981	-0.6	-2.7	-1.4	-13.6	-23.6	-9.1	-23.9	-23.2
1982	-0.1	-3.9	-1.5	-14.0	-26.8	-5.8	-20.6	-22.7

Table 3. (continued)

Year	Y	D	SR	I	K
		Non-recursive type by 2SLS			
1965	-0.0	-0.0	-0.0	-0.1	-0.0
1966	0.7	4.7	2.4	3.0	0.4
1967	0.4	7.1	-2.7	-2.3	-0.6
1968	-2.5	8.4	-5.5	-8.4	-3.0
1969	-5.8	9.3	-5.9	-12.2	-5.7
1970	-7.7	8.9	-10.1	-18.7	-9.3
1971	-5.0	11.6	-2.2	-7.4	-10.2
1972	-6.7	9.5	-1.5	-8.4	-11.0
1973	-9.2	5.0	-5.1	-14.8	-12.7
1974	-1.9	2.0	2.1	0.2	-11.4
1975	-0.8	6.0	-23.1	-24.0	-14.0
1976	-3.0	2.0	-18.9	-22.4	-16.0
1977	-5.2	-4.6	-15.9	-22.0	-17.4
1978	-7.4	-8.6	-17.5	-26.2	-19.0
1979	-9.7	-12.8	-17.8	-29.3	-20.6
1980	-11.5	-16.8	-11.1	-23.9	-21.3
1981	-12.9	-21.7	-8.3	-22.3	-21.4
1982	-13.2	-24.9	-4.6	-18.4	-20.8
		Non-recursive type by OLS			
1965	-0.0	-0.0	-0.0	-0.1	-0.0
1966	0.6	4.7	1.5	2.1	0.2
1967	0.1	6.9	-4.3	-4.5	-1.3
1968	-3.0	7.8	-7.7	-11.1	-4.1
1969	-6.5	8.4	-8.5	-15.7	-7.4
1970	-8.7	7.6	-13.0	-22.9	-11.4
1971	-6.1	10.0	-5.1	-11.5	-12.7
1972	-8.0	7.5	-4.4	-12.8	-14.0
1973	-10.7	2.6	-8.0	-19.6	-16.0
1974	-3.3	-0.7	-0.6	-3.9	-14.9
1975	-2.0	3.2	-22.1	-24.6	-17.2
1976	-4.1	-0.6	-18.3	-23.2	-19.0
1977	-6.3	-7.1	-15.7	-23.0	-20.2
1978	-8.4	-11.1	-17.7	-27.6	-21.6
1979	-10.7	-15.2	-18.3	-31.0	-23.2
1980	-12.5	-19.2	-11.9	-25.9	-23.7
1981	-13.9	-24.1	-9.4	-24.6	-23.8
1982	-14.1	-27.3	-5.9	-20.9	-23.2

A few points of interest can emerge from these tables. First of all, in the recursive case, the labor force component represented by LM, LF and L indicates a fully satisfactory level of traceability, but the capital side shows considerably pronounced deviations. The main source of these deviations is attributable to the relatively limited performance of the saving rate equation. Because of the nature of this model, the simulation error for savings is accumulated in the simulated value of capital stock over time, so that the simulatability of aggregate output gradually diminishes over the simulation period. This result is applicable to the two other cases as well.

Secondly, although these three cases have yielded somewhat different simulated values, the non-recursive type by 2SLS seems to trace the variation in real GNP and capital-related variables more closely than the two other types. It should be stressed, however, that the differences in the traceability of these three cases are very small, and vary considerably from year to year. For instance, the non-recursive type by 2SLS shows the least deviation among these three cases in the final simulation year, but in some other years the recursive type or the non-recursive type by OLS traces the actual variation in a more satisfactory way.

Thirdly, the recursive type and the non-recursive type by OLS show a highly comparable pattern of changes in the simulated values. In the first half of the simulation period, the simulated values of the key variables for the latter type are closer to the actual values than those for the former type. In the second half of the simulation period, however, the opposite pattern holds.

In any case, the results of the final test of these three models seem to suggest that because the simulated values are different among the three models only to a marginal extent, none of these models are superior to the others in terms of simulatability. Although this conclusion may not be applicable to the general case, it appears to imply that in some cases the simultaneity bias may not give rise to any serious difficulties in simulation exercises. In fact, one of the recent studies conducted by the Economic Planning Agency (EPA) of the Government of Japan has demonstrated on an empirical basis that no significant difference was detected in the traceability of the large-scale non-recursive models for Japan and the West Germany estimated by OLS, 2SLS, LIML (Ito, et al., 1984).

Moreover, even if non-recursive models estimated by OLS and those by 2SLS would produce differences of noticeable magnitude in terms of simulated results, the possibility of using OLS in estimating the structural form coefficients of large-scale econometric demographic models should not be totally eliminated partly because it is the simplest statistical technique, and partly because such models are usually designed not for short-run predictions but for policy-oriented numerical experiments with a time horizon of ten years or more.

This view is further endorsed by projected results of the three different types of the model over the period 1982-2000, as displayed in Table 4. As regards real GNP, the non-recursive type by OLS shows the largest value throughout the entire projection period, while the recursive type the smallest value. The difference between these values, however, amounts to only 3.5 percent in the final year of the projection period. The difference of such a small magnitude does not seem to influence any policy conclusions to be derived from simulation exercises. It is also interesting to note that the EPA study cited in the above has found that in terms of the projected results beyond the sample period, the non-recursive model estimated by OLS predicted the variation of major economic indicators more accurately than that by 2SLS or LIML.

Econometricians often employ sophisticated techniques to improve the goodness of fit. One of the salient examples of such advanced statistical methods is the spline technique (Suits, Mason, and Chan). Some of the equations estimated by this technique, however, pose serious problems when they are used for projection exercises. These estimated equations could be extremely powerful in the analysis of policy alternatives within a certain sample period (Ogawa and Suits), but they could yield unreasonable results in the case of extrapolation due to their non-linear nature. Similar problems tend to occur with regard to the use of piece-wise regression equations.

III. Two Point Iterative Methods

In this section, we describe two most representative methods in which the approximate solution is improved point by point. We also present here, without proof, rules and suggestions for their efficient use.

Table 4. Simulated Results for Three Different Types of the Model, 1983-2000

Year	LM	LF	L	Y	D	SR	I	K
				Recursive type				
1983	35,625	22,646	58,271	209,189	39,438	0.327	68,479	612,207
1984	36,056	22,807	58,863	214,659	41,295	0.331	71,124	642,037
1985	36,451	22,924	59,375	219,991	43,202	0.335	73,699	672,533
1986	36,886	23,063	59,949	225,475	45,152	0.339	76,363	703,744
1987	37,322	23,205	60,527	230,994	47,148	0.342	79,045	735,641
1988	37,751	23,349	61,100	236,524	49,187	0.346	81,727	768,181
1989	38,109	23,477	61,586	241,869	51,268	0.349	84,316	801,229
1990	38,358	23,565	61,923	246,857	53,381	0.351	86,731	834,578
1991	38,544	23,628	62,171	251,593	55,514	0.354	89,016	868,080
1992	38,679	23,675	62,354	256,108	57,656	0.356	91,197	901,621
1993	38,769	23,703	62,473	260,392	59,801	0.358	93,263	935,082
1994	38,827	23,716	62,542	264,467	61,941	0.360	95,223	968,365
1995	38,849	23,717	62,566	268,327	64,069	0.362	97,073	1,001,369
1996	38,822	23,694	62,516	271,891	66,179	0.363	98,777	1,033,966
1997	38,766	23,658	62,424	275,227	68,263	0.365	100,365	1,066,066
1998	38,691	23,610	62,301	278,356	70,316	0.366	101,846	1,097,594
1999	38,610	23,561	62,171	281,336	72,332	0.367	103,261	1,128,522
2000	38,496	23,495	61,991	284,054	74,310	0.368	104,541	1,158,753

Year	Y	D	SR	I	K
		Non-recursive type by 2SLS			
1983	209,522	39,436	0.329	68,856	612,554
1984	215,370	41,317	0.334	71,885	643,114
1985	221,078	43,271	0.338	74,592	674,459
1986	227,000	45,275	0.343	77,648	706,847
1987	233,008	47,346	0.347	80,741	740,256
1988	239,067	49,483	0.351	83,842	774,629
1989	244,981	51,681	0.355	86,855	809,816
1990	250,569	53,930	0.359	89,694	845,590
1991	255,921	56,218	0.362	92,391	881,773
1992	261,067	58,532	0.364	94,974	918,224
1993	265,889	60,863	0.367	97,439	954,797
1994	270,623	63,201	0.369	99,773	991,385
1995	275,146	65,541	0.371	101,991	1,027,866
1996	279,378	67,873	0.373	104,053	1,064,101
1997	283,374	70,190	0.374	105,988	1,099,965
1998	287,154	72,484	0.376	107,805	1,135,361
1999	290,776	74,747	0.377	109,542	1,170,229
2000	294,129	76,977	0.378	111,137	1,204,476

Table 4. (continued)

Year	Y	D	SR	I	K
		Non-recursive type by OLS			
1983	209,441	39,436	0.327	68,391	612,116
1984	215,191	41,289	0.332	71,198	642,040
1985	220,843	43,202	0.336	73,957	672,809
1986	226,667	45,170	0.340	76,808	704,461
1987	232,555	47,194	0.343	79,693	736,973
1988	238,479	49,273	0.347	82,590	770,303
1989	244,245	51,404	0.350	85,409	804,320
1990	249,679	53,579	0.353	88,070	838,820
1991	254,873	55,785	0.356	90,602	873,646
1992	259,748	58,012	0.358	93,041	908,647
1993	264,525	60,250	0.361	95,352	943,747
1994	269,100	62,495	0.363	97,564	978,832
1995	273,469	64,738	0.365	99,668	1,013,792
1996	277,553	66,974	0.366	101,628	1,048,502
1997	281,407	69,193	0.368	103,472	1,082,848
1998	285,052	71,389	0.369	105,206	1,116,740
1999	288,546	73,556	0.371	106,870	1,150,128
2000	291,778	75,691	0.372	108,399	1,182,925

Perhaps, the simplest of the point iterative methods is the Gauss-Seidel method, also known as the method of successive displacements. Because of its simplicity, the Gauss-Seidel method is most widely employed to solve simultaneous linear equations models. Application of this method to the non-linear system has been also demonstrated at theoretical levels by Bers (1953), Schechter (1962), etc. The other point iterative method, which is similar and equally simple, is the Jacobi method or the method of simultaneous displacements. Although there are many more iterative methods for solving large-scale econometric models such as the Newton method and the Simplified Newton method, we confine our discussions to both Gauss-Seidel and Jacobi methods, using some illustrative examples.

Let us now compare methodological differences between these two methods. To do so, we consider a simultaneous linear equations model as follows:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \qquad \text{(III-I)}$$

$$\cdots\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

This system of equations can be re-arranged as below:

$$x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2^0 - a_{13}x_3^0 - \cdots - a_{1n}x_n^0)$$

$$x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1^1 - a_{23}x_3^0 - \cdots - a_{2n}x_n^0)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(III-2)}$$

$$x_3^1 = \frac{1}{a_{33}}(b_3 - a_{31}x_1^1 - a_{32}x_2^1 - \cdots - a_{3n}x_n^0)$$

$$\cdots\cdots$$

$$x_n^1 = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^1 - a_{n2}x_2^1 - \cdots - a_{nn-1}x_{n-1}^1)$$

In the Gauss-Seidel method, starting with arbitrary initial values x_2^0, x_3^0, . . . , x_n^0, one improves the value for x_1^1, and this improved value is used along with x_3^0, . . . , x_n^0 to compute x_2^1 in the second equation shown by III-2. For the third equation, these improved values x_1^1 and x_2^1 are used with x_3^0, . . . , x_n^0 to calculate x_3^1. The same computational procedure is applied to the rest of equations. Mathematically, this approximation process can be generalized as follows:

$$x_1^{k+1} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^k - a_{13}x_3^k - \cdots - a_{1n}x_n^k)$$

$$x_2^{k+1} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{k+1} - a_{23}x_3^k - \cdots - a_{2n}x_n^k)$$

$$x_3^{k+1} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{k+1} - a_{32}x_2^{k+1} - a_{34}x_4^k - \cdots - a_{3n}x_n^k)$$

$$\cdots\cdots$$

$$x_n^{k+1} = \frac{1}{a_{nn}} (b_n - a_{n1}x_1^{k+1} - a_{n2}x_2^{k+1} - \ldots - a_{nn-1}x_{n-1}^{k+1})$$

<div align="right">(III-3)</div>

In the case of the Jacobi method, however, one does not use improved values until after a complete iteration. The improvement formula for this method is:

$$x_1^{k+1} = \frac{1}{a_{11}} (b_1 - a_{12}x_2^k - a_{13}x_3^k - \ldots - a_{1n}x_n^k)$$

$$x_2^{k+1} = \frac{1}{a_{22}} (b_2 - a_{21}x_1^k - a_{23}x_3^k - \ldots - a_{2n}x_n^k)$$

$$x_3^{k+1} = \frac{1}{a_{33}} (b_3 - a_{31}x_1^k - a_{32}x_2^k - \ldots - a_{3n}x_n^k) \qquad \text{(III-4)}$$

$$\cdot \; \cdot \; \cdot \; \cdot \; \cdot$$

$$x_n^{k+1} = \frac{1}{a_{nn}} (b_n - a_{n1}x_1^k - a_{n2}x_2^k - \ldots - a_{nn-1}x_{n-1}^k)$$

In either method, a sufficient condition for convergence is expressed as follows:

$$\max_i \left(\frac{1}{|a_{ii}|} \cdot \Sigma |a_{ij}| \right) < 1, \qquad i = 1, 2, \ldots, n \qquad \text{(III-5)}$$

Because the proof of this theorem is available elsewhere (Todd, 1962), no explanation on the derivation of this condition is needed here. It should be noted, however, that because this is not a necessary condition, the system could still possibly converge in the Gauss-Seidel or Jacobi method even though the condition (III-5) is not satisfied. Furthermore, it is conceivable that certain systems could be solved by the Gauss-Seidal method, and not by the Jacobi method. The opposite case is also conceivable as shown by the following example. Suppose we have two variables, HA representing hours worked by the average worker per month and HP potential hours worked by the average workers per month. By definition, HP is greater than HA. The relationship between these variables can be specified as below:

> ln HA = a + b ln (WAGE/CPD)
> HP = HA + constant

where WAGE: annual wage per worker

CPD: personal consumption deflator.

For time t and the k+1-th iteration by the Jacobi method, we have

$$\ln HA_t^{k+1} = a + b \ln (WAGE_t^k/CPD_t^k)$$

$$HP_t^{k+1} = HA_t^k + constant.$$

The problem lies in the relationship between HA_t^{k+1} and HA_t^k. When the value of HA_t^k takes a very small value in the process of convergence, HP_t^{k+1} viz. HA_t^k + constant is smaller than HA_t^{k+1}. This case violates the definition. If we use the Gauss-Seidel method, this problem can be avoided.

It should be noted, however, that there are cases in which neither method can solve systems. Suppose that we have the following simple system of equations:

$$Y = f (X)$$
$$X = g (Y) \qquad\qquad\qquad (III-6)$$

If these two equations had the relationship as expressed by Panel A or Panel B of Figure 1, the system would never converge, no matter which method is utilized. In the case of Panel A, every other round of iteration gives the same set of values for X and Y. For Panel B, the system explodes rather than converges.

Let us now compare computational results of these iterative methods, by applying them to our non-recursive model estimated by 2SLS. As regards the Jacobi method, the results have already been presented in Table 3 and discussed in detail in the previous section. For this reason, we have shown only the results by the Gauss-Seidel method in Tables 5 and 6. Table 5 presents the performance of the model by the final test, while Table 6, the projected results over the period 1983-2000. A brief examination of Tables 3, 5 and 6 reveals that these two iterative methods have produced highly comparable results.

It is, however, of importance to note that the rate of convergence is substantially different between these two methods. It has been mathematically proven that the rate of convergence of the

- 15 -

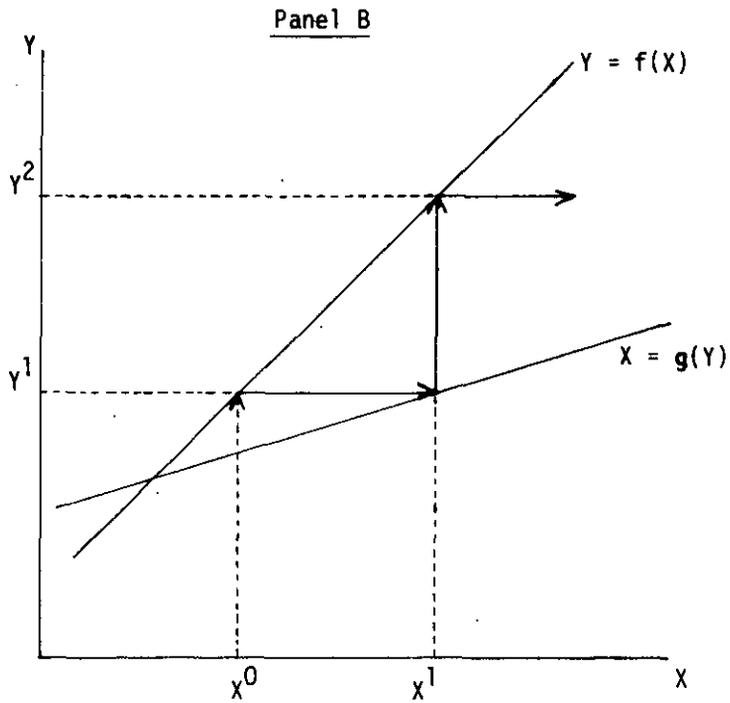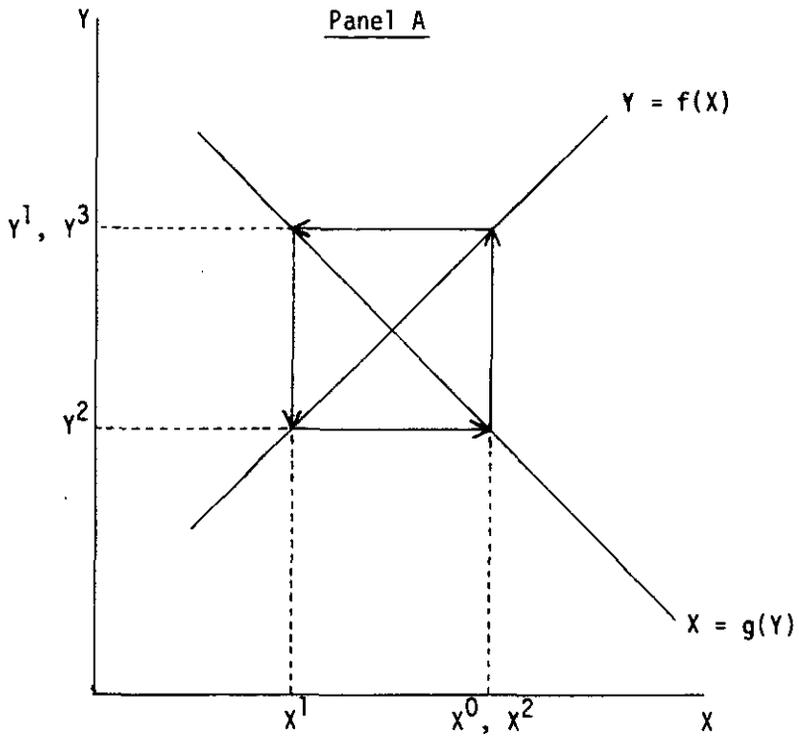Figure 1. Illustration of No Convergency

Panel A

$Y = f(X)$

Y^1, Y^3

Y^2

$X = g(Y)$

X^1 X^0, X^2 X

Panel B

$Y = f(X)$

Y^2

Y^1

$X = g(Y)$

X^0 X^1 X

Table 5. Performance of Non-recursive Model Estimated by 2SLS
and Solved by the Gauss-Seidel Method

(Unit: %)

Year	Y	D	SR	I	K
1965	0.0	-0.0	0.0	0.0	0.0
1966	0.7	4.7	2.4	3.1	0.5
1967	0.4	7.2	-2.6	-2.2	-0.5
1968	-2.4	8.5	-5.4	-8.0	-2.8
1969	-5.7	9.4	-5.8	-11.9	-5.5
1970	-7.6	9.0	-10.0	-18.4	-9.0
1971	-4.9	11.8	-2.2	-7.1	-9.9
1972	-6.6	9.7	-1.5	-8.2	-10.7
1973	-9.1	5.2	-5.0	-14.6	-12.4
1974	-1.8	2.3	2.1	0.4	-11.1
1975	-0.6	6.2	-23.0	-23.8	-13.7
1976	-2.8	2.3	-18.7	-22.0	-15.7
1977	-5.1	-4.3	-15.7	-21.6	-17.1
1978	-7.2	-8.3	-17.3	-25.8	-18.7
1979	-9.5	-12.5	-17.6	-28.9	-20.3
1980	-11.4	-16.5	-10.9	-23.5	-20.9
1981	-12.7	-21.4	-8.2	-21.9	-21.0
1982	-13.0	-24.5	-4.5	-18.0	-20.4

Table 6. Simulated Results for Non-recursive Model Estimated by
2SLS and Solved by the Gauss-Seidel Method

Year	Y	D	SR	I	K
1983	209,553	39,438	0.329	68,893	612,627
1984	215,408	41,321	0.334	71,924	643,230
1985	221,174	43,278	0.339	74,896	674,848
1986	227,142	45,300	0.343	77,979	707,527
1987	233,188	47,390	0.348	81,087	741,224
1988	239,283	49,544	0.352	84,200	775,879
1989	245,223	51,760	0.356	87,214	811,333
1990	250,829	54,028	0.359	90,046	847,351
1991	256,201	56,331	0.362	92,737	883,758
1992	261,365	58,659	0.365	95,314	920,413
1993	266,305	61,003	0.367	97,765	957,175
1994	271,039	63,353	0.369	100,098	993,920
1995	275,559	65,703	0.371	102,307	1,030,524
1996	279,779	68,043	0.373	104,358	1,066,837
1997	283,764	70,365	0.375	106,281	1,102,751
1998	287,532	72,662	0.376	108,085	1,138,173
1999	291,132	74,927	0.377	109,805	1,173,051
2000	294,468	77,157	0.378	111,386	1,207,279

Gauss-Seidel method is just twice that of the Jacobi method (Todd, 1962, p. 404). It can be seen that if a system is solvable by either method, the gain in the rate of convergence is very large in cases where the Gauss-Seidel method is slow or where the system is extremely huge.

It is also well-known that in the case of the Gauss-Seidel method, the rate of convergence is substantially affected by the ordering of equations to be solved. The results in Tables 5 and 6 have been computed on the basis of the order of equations listed on pages 2-3. Suppose that we move the first three equations to the bottom of the equation list. In the case of this new ordering, the converged values are almost the same as those in the case of the original ordering, but the speed of convergence is considerably slower in the former case, as displayed in Table 7. Although the differences in the rate of convergence between these two cases vary from year to year, the former is, on average, more than twice as fast as the latter.

Another factor which affects the rate of convergence in these two iterative methods is the criterion of convergence. The use of a severer criterion leads to an increase in the number of iterations at an accelerating rate. In the present analysis, we have set the criterion at a 0.1 percent level. When we have shifted the criterion to a 0.01 percent level, the number of iterations for our non-recursive model estimated by 2SLS has increased roughly by 25 percent under the Gauss-Seidel method and by 50 percent under the Jacobi method.

Table 7. Comparison of the Rate of Convergence Between Two Cases

Year	Number of iterations	
	Original ordering	New ordering
1965	2	2
1966	5	12
1967	5	11
1968	5	11
1969	5	10
1970	5	9
1971	5	8
1972	4	9
1973	4	8
1974	4	8
1975	4	12
1976	4	9
1977	4	9
1978	4	9
1979	4	9
1980	4	9
1981	4	9
1982	4	9

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