Method of Computing the Expectation of Life at Old Age on the Basis of the Principle of Agreement with Data

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ABSTRACT

Methods of constructing abridged life tables have been studied for many years. In regard to old age, however, it is hard to claim that we have found a sufficiently adequate method of calculating values of life expectancy. This paper presents a method of calculating values of life expectancy at ages 80 or more, on the basis of the principle of agreement with the data. Regarding the availability of the base data of population and deaths for life table construction, we deal with the following three cases. The first is that the population data are given in five-year age groups up to the maximum age and the deaths in five-year age groups to age z (z = 80, 85 or 90) and open-ended beyond this age. The second is that the former are given in five-year age groups up to age z (z = 80, 85 or 90), open-ended beyond this age; and the latter in five-year age groups up to the maximum age. The third case explores both of the population and deaths in five-year age groups up to age z (z = 80, 85 or 90) and open-ended beyond this age. Use of a Keyfitz and Frauenthal's formula for obtaining life table q_x values is introduced in the present paper and this is different from the technique presented in our previous paper (1984) which was also based on the principle of agreement with the data.

I. Introduction

It could be generally stated in the life table that the share of stationary population or total after lifetime (T_x) in old age becomes greater as mortality becomes lower. In preparing a life table with lower mortality, consequently, the method of calculating values of expectation of life at old age becomes more sensitive to the values of expectation of life at younger ages. In spite of a three-century long history of research in modern methods of constructing life tables, less attention has been paid to elaborating techniques on how to calculate as precisely as possible values of expectation of life at old age. This paper presents a method of calculating these values on the basis of the principle of agreement with the data. The same subject was dealt with in our previous paper (Kobayashi and Nanjo, 1984); but we demonstrate a new method in the present paper. It is expected that the values of $\mathring{e}_t$ computed by our method are approximately equal to the ones of $\mathring{e}_t$ computed by a fairly precise method of making a life table (e.g. Keyfitz, 1968; 1977) when both population and deaths are classified by 5-year age groups up to 100 years of age.

Let the mid-year population, deaths and central death rate in the age group (x, x+5) be denoted by ${}_5K_x$, ${}_5D_x$, and ${}_5M_x$ (= ${}_5D_x/{}_5K_x$) respectively. Usual notations of life table functions are used here.

Makeham's curve has been commonly used to obtain the fairly precise values of expectation of life at very old ages.

By extrapolation on the basis of Makeham's curve passing through the four points, namely number of surviving l_{z-15}, l_{z-10}, l_{z-5} and l_z at ages z-15, z-10, z-5 and z respectively, the values of l_{z+5}, l_{z+10},... are computed, and then other life table functions, that is, ${}_5L_x$, T_x, and $\mathring{e}_x$ are easily computed. For computing the values of ${}_5L_x$, a formula of integrating l_x numerically is applied.

But the values of $\mathring{e}_z$ ($z \geq 80$) computed in this way greatly depend upon some of the values of l_{z-15}, l_{z-10}, l_{z-5} and l_z, especially upon that of l_z, so Keyfitz (1968) used the method based on the principle of agreement with the data. He assumed that

$$M_{80+t} = {}_5M_{80} (1 + \beta t) \text{ for } t = 0, 5, 10, \ldots .$$

In his expression, it may be supposed that the ${}_5M_x$ beyond age 85 rise not linearly but as the exponential of x.

In this paper, we show a useful approach for the computation of values of $\mathring{e}_z$ ($z \geq 80$) on the basis of the principle of agreement with the data using Makeham's curve.

The population and deaths used here were classified by five-year age groups and for the last age group the following cases were discussed:

Case 1. the last age group of deaths is z years and over,

Case 2. the last age group of population is z years and over,

Case 3. the last age groups of population and deaths are both z years and over,

where z = 80, 85 and 90.

In the following, z stands for the value of the last age group: z years of age and over as mentioned above.

II. Preliminaries

A.- C. were applied to all the cases and D. was used only for Case 3.

A. Some formulae to compute the probability of surviving from age x to x+5 (${}_5p_x$)

Combined use is made of the following formulae which compute the probability of surviving from age x to x + 5 (${}_5p_x$) from the central death rate (${}_5M_x$) (not the life table mortality rate ${}_5m_x$) in the age group (x, x+4).

i. Reed-Merrell's formula

$$ {}_5p_x = \exp\left(-5 \cdot {}_5M_x - 125 \cdot A_x \cdot {}_5M_x^{\,2}\right) \tag{1} $$

with $A_x = 0.008$ (x = 5, 10, 15,...). (See Reed and Merrell, 1939).

ii. Keyfitz-Frauenthal proposed the formula

$$ {}_5p_x = \exp\left\{ -5 \cdot {}_5M_x + \frac{5}{48 \cdot {}_5K_x} ({}_5K_{x+5} - {}_5K_{x-5})({}_5M_{x+5} - {}_5M_{x-5}) \right\} \tag{2} $$

which gives fairly precise reults. (Keyfitz & Frauenthal,1975; Keyfitz, 1977)

iii. Greville also demonstrated the formula

$$ {}_5p_x = \exp\left\{ -5 \cdot {}_5M_x + (5/48) \cdot ({}_5M_{x+5} - {}_5M_{x-5}) \cdot \ln({}_5K_{x+5}/{}_5K_{x-5}) \right\} \tag{3} $$

which gives results very similar to the ones obtained by the formula (2). (See Keyfitz et al. 1975)

B. Computation of the values of A_x

From (1) and (2) in II.A. we have approximately

$$-125 \cdot A_x \cdot {}_5M_x^{\,2} = \frac{5}{48 \cdot {}_5K_x}({}_5K_{x+5} - {}_5K_{x-5})({}_5M_{x+5} - {}_5M_{x-5})$$

or

$$A_x = -({}_5K_{x+5} - {}_5K_{x-5})({}_5M_{x+5} - {}_5M_{x-5})/(1200 \cdot {}_5K_x \cdot {}_5M_x^{\,2}). \qquad (4)$$

The suitable values of A_x for $x \leqq z - 10$ are obtainable by using the formula (4) from the given data, but those for $x > z - 10$ have to be estimated by another method. Now on the basis of the values of A_x computed from the 1975 and 1980 data for males and females in Japan, we obtained the approximation

$$A_x = 0.01 \times 1.2^{(90-x)/5} \qquad (5)$$

for $x \geqq 70$.

The values of A_x can vary with country, time and sex. The formula (5) can be slightly corrected on the basis of the values of ..., A_{z-20}, A_{z-15}, and A_{z-10} computed using (4) in advance from the given data. The more suitable the values of A_x for $x > z-10$ are, the more precise our results become.

For reference, let us consider the values of A_x computed by (4) using the mid-year population and deaths for Japanese women, 1980, which were used in a previous paper (Kobayashi and Nanjo, 1984).

Ages x	...	60	65	70	75	80	85	90
Values of A_x	...	.0399	.0286	.0275	.0202	.0153	.0124	.0108

C. Computation of the values of ${}_5M_x$

From (1), we have

$${}_5M_x = (-1 + \sqrt{1 - 20 \cdot A_x \cdot \ln {}_5p_x})\,/(50 \cdot A_x), \qquad (6)$$

where A_x are the values obtained in II, B.

There can be some formulae of obtaining the values of ${}_5M_x$ other than (6), but (6) seems to give us good results in using our iteration procedure (cf. III).

D. Estimation of the values of l_x at old ages

To estimate the values of l_x at old ages, the following formula

$$u_x = 4u_x - 6u_{x-5} + 4u_{x-10} - u_{x-15} \tag{7}$$

with $u_x = \ln l_x$, will be used instead of Makeham's curve in IV (e.g. M. Spiegelman, 1968).

E. Estimation of population ${}_5K_z$, ${}_5K_{z+5}$, ...

Here is shown the method of estimating (tentative) populations ${}_5K_z$, ${}_5K_{z+5}$, ... from both the given data on population and deaths classified by five-year age groups at ages under z years and (tentatively) given values of l_z, l_{z+5}, ...

The values of the probability ${}_5p_z$, ${}_5p_{z+5}$, ... were obtained by

$${}_5p_t = l_{t+5}/l_t$$

for t = z, z+5, ..., and then the values of ${}_5M_z$, ${}_5M_{z+5}$, ... were computed by (6) in II.C.

Now, from (1) and (3), we have approximately

$$-125 \cdot A_x \cdot {}_5M_x^{\,2} = (5/48)\cdot({}_5M_{x+5} - {}_5M_{x-5})\cdot \ln({}_5K_{x+5} / {}_5K_{x-5}),$$

or, putting $cp\,(x) = {}_5K_{x+5} / {}_5K_{x-5}$,

$$cp\,(x) = \exp\{1200\cdot A_x\cdot {}_5M_x^{\,2} / ({}_5M_{x-5} - {}_5M_{x+5})\}. \tag{8}$$

Then (8) and

$${}_5K_{x+5} = {}_5K_{x-5} \cdot cp(x) \tag{9}$$

are the formulae which provide the value of ${}_5K_{x+5}$ from the given values of ${}_5M_{x-5}$, ${}_5M_x$, ${}_5M_{x+5}$, A_x and ${}_5K_{x-5}$.

If the given data are

$$\cdots,\ {}_5K_{50},\ {}_5K_{55},\ \cdots,\ {}_5K_{z-5},\ {}_\infty K_z$$

and

$$\cdots,\ {}_5D_{50},\ {}_5D_{55},\ \cdots,\ {}_5D_{z-5},\ {}_\infty D_z$$

for z = 80, 85 and 90, then, using (8) and (9), population ${}_5K_z$ can be obtained from the values of ${}_5M_{z-10}$, ${}_5M_{z-5}$, ${}_5M_z$, A_{z-5} and ${}_5K_{z-10}$, and population ${}_5K_{z+5}$ can be obtained from the values of ${}_5M_{z-5}$, ${}_5M_z$, ${}_5M_{z+5}$, A_z and ${}_5K_{z-5}$.

In other words, tentative values of ${}_5K_z$, ${}_5K_{z+5}$, ... can be obtained from the given population and deaths under z years and tentatively given values of l_z, l_{z+5}, ...

Populations by five-year age groups at old ages which are obtained using (8) and (9) are not very precise but effectual enough for the present study.

III. The Method of Computing the Values of $\mathring{e}_z$ at Old Ages

To begin with, the methods for the Cases 1 and 2 are explained.

A. Case 1. Data are as follows:

Population: ..., ${}_5K_{55}$, ... , ${}_5K_{z-5}$, ${}_5K_z$, ... , ${}_5K_{95}$, ${}_\infty K_{100}$

Deaths : ...., ${}_5D_{55}$, ... , ${}_5D_{z-5}$, ${}_\infty D_z$

where z = 80, 85 and 90.

From the given data, the values of ..., ${}_5M_{55} = {}_5D_{55} / {}_5K_{55}$, ${}_5M_{60} = {}_5D_{60} / {}_5K_{60}$, ... , ${}_5M_{z-5} = {}_5D_{z-5} / {}_5K_{z-5}$ were computed and from these values, the values of the probability, ..., ${}_5p_{55}$, ..., ${}_5p_{z-10}$ were obtained using Keyfitz-Frauenthal's formula (2) and the value of ${}_5p_{z-5}$ was obtained using the formula (1) with the value of A_{z-5} (in the formula (5)). From these values of ${}_5P_x$ for $x \leq z-5$ the values of ..., l_{60}, l_{65},...,l_z were computed using the formula $l_{t+5} = l_t \cdot {}_5p_t$.

To obtain the values of $\mathring{e}_t$ for $t \geq z$, the value of l'_{z+5} which is a tentative value of l_{z+5} was used here.

By fitting Makeham's curve to the four values of l_{z-10}, l_{z-5}, l_z and l_{z+5} (l'_{z+5} was denoted by l_{z+5}), the values of l_{z+10}, l_{z+15}, ... were estimated. From these values of l_t, the values of ${}_5p_t = l_{t+5}/l_t$ were computed and then the values of ${}_5M_t$ for t = z, z+5,... were obtained from (6) with the values of A_t in the approximation formula (5).

Now

$$\sum_{t=z}^{95} {}_5K_t \cdot {}_5M_t = {}_\infty D'_z, \text{ say.} \tag{10}$$

Then ${}_\infty D_z$ and ${}_\infty D'_z$ are the given deaths and the computed deaths (the expected deaths), in each iteration at ages z and over, respectively.

The difference $y = {}_{\infty}D'_z - {}_{\infty}D_z$ depends upon the value of l'_{z+5}, that is, it is a function of l'_{z+5}. So let it be denoted by

$$y = f(l'_{z+5}). \tag{11}$$

The regula falsi was used to get the value of l'_{z+5} that satisfies $f(l'_{z+5}) = 0$ approximately. Iteration procedures were continued until $|f(l'_{z+5})| \leqq {}_{\infty}D_z \times 10^{-5}$ held. Makeham's curve fitted at the final iteration was used to get the final values of l_t for $t > z$. Then the values of $\mathring{e}_t$ for $t \geqq z$ were obtained by applying the formulae

$$_1L_t = (13/24) \cdot (l_t + l_{t+1}) - (1/24) \cdot (l_{t-1} + l_{t+2})$$

and

$$\sum_{i=0}^{109} {}_1L_{t+i}/l_t = \mathring{e}_t.$$

B. Case 2. The data used here are

Population: $\ldots, {}_5K_{55}, \ldots, {}_5K_{z-5}, {}_{\infty}K_z$

Deaths : $\ldots, {}_5D_{55}, \ldots, {}_5D_{z-5}, {}_5D_z, \ldots, {}_5D_{95}, {}_{\infty}D_{100}$

where z = 80, 85 and 90.

The method for Case 2 is the same as that for Case 1 except that

$${}_{\infty}K'_z = \sum_{z}^{95} {}_5D_t \,/\, {}_5M_t$$

was used instead of (10), and

$$y = {}_{\infty}K'_z - {}_{\infty}K_z = f(l'_{z+5})$$

was used instead of (11).

C. Case 3. The data used are

Population: $\ldots, {}_5K_{55}, \ldots, {}_5K_{z-5}, {}_{\infty}K_z$

Deaths : $\ldots, {}_5D_{55}, \ldots, {}_5D_{z-5}, {}_{\infty}D_z$

where z=80, 85 and 90.

The method for Case 3 is almost the same as that for Case 1 except that a set of tentative values of population ${}_5K_t$ (t=z, z+5,...) has to be estimated in each iteration.

More precisely speaking, Makeham's curve fitted to the four values, that is, the values of l_{z-10}, l_{z-5} and l_z obtained from the data as in III.A. and l'_{z+5}, the value tentatively used in each iteration in the regula falsi provided the tentatively estimated values of l_t for $t > z$.

From these values of l_t and the data above mentioned, a set of tentatively estimated values of ${}_5K_z$, ${}_5K_{z+5}$,... were computed using the method explained in II.E.

IV. Numerical Illustration

The data prepared for numerical illustration are age-specific mid-year population and deaths for Japanese males and females in 1980.

For computation of values of $\mathring{e}_z$ were used the data on population and deaths by five-year age groups whose last age groups are indicated in the Cases 1, 2 and 3 in III. above.

In the following tables, the values of $\mathring{e}_t$ $(t \geq z)$ computed are denoted by $\mathring{e}_t\ (z)$ when the last age group is $z \leq t < \infty$.

Makeham's curve was used in Table 1 and the formula (7) was used in Table 2. The standard values of $\mathring{e}_z$ to be compared with the results obtained were computed on the basis of the values of ${}_5p_z$ for z=...,75, 80, 85,... obtained using the formula (2) (the last column in the Tables 1 and 2). To get the value of T_{100}, the l_x curve extrapolated by the Makeham formula was used.

In Table 1, the values of $\mathring{e}_{85}(85)$ for males are 4.37, 4.36 and 4.34 in the Cases 1, 2, and 3, respectively. These are very close to the standard values of $\mathring{e}_{85}$, 4.38. On the other hand, the values of $\mathring{e}_{90}(85)$ are 3.11, 3.10 and 3.06 in the Cases 1, 2, and 3, respectively, and the values of $\mathring{e}_{95}$ (85) are 2.19, 2.18 and 2.13 in the Cases 1, 2, and 3, respectively. These values of $\mathring{e}_{90}$ (85) and $\mathring{e}_{95}$ (85) are slightly different from the standard values, 3.15 and 2.20 for t= 85 and 90, respectively.

From Tables 1 and 2, the following results were obtained.

1) In each case, precision in computation of the values of $\mathring{e}_t$ (z) for $t \geq z$ tends to increase as age z advances.
2) The results obtained in the Cases 1 and 2 are as a matter of course better than that in Case 3.
3) The values of $\mathring{e}_t$ (z) at very old ages $t > z$ computed using Makeham's curve tend to be smaller than those computed using the formula (7).
4) Our values of $\mathring{e}_z(z)$ are very close to those obtained in Kobayashi

Table 1. Values of $\mathring{e}_t(z)$ $(t \geq z)$ for z=80, 85, 90 in Cases 1, 2 and 3 When the Last Age Group of the Data is $z \leq t < \infty$, Japan, 1980 (using Makeham's Curve)

$\mathring{e}_t(z)$ for $t \geq z$		Case 1			Case 2			Case 3			Standard values of $\mathring{e}_t$
		Values of z			Values of z			Values of z			
		80	85	90	80	85	90	80	85	90	
Males	$\mathring{e}_{80}(z)$	6.06	----	----	6.03	----	----	5.99	----	----	6.08
	$\mathring{e}_{85}(z)$	4.31	4.37	----	4.28	4.36	----	4.23	4.34	----	4.38
	$\mathring{e}_{90}(z)$	3.01	3.11	3.15	2.97	3.10	3.15	2.92	3.06	3.15	3.15
	$\mathring{e}_{95}(z)$	2.07	2.19	2.26	2.02	2.18	2.26	1.97	2.13	2.26	2.20
Females	$\mathring{e}_{80}(z)$	7.39	----	----	7.30	----	----	7.64	----	----	7.34
	$\mathring{e}_{85}(z)$	5.10	5.17	----	5.00	5.15	----	5.41	5.16	----	5.15
	$\mathring{e}_{90}(z)$	3.39	3.57	3.60	3.27	3.54	3.59	3.71	3.55	3.60	3.57
	$\mathring{e}_{95}(z)$	2.17	2.42	2.50	2.06	2.39	2.49	2.48	2.40	2.51	2.49

Table 2. Values of $\mathring{e}_t(z)$ $(t \geq z)$ for z=80, 85, 90 in Cases 1, 2 and 3 When the Last Age Group of the Data is $z \leq t < \infty$, Japanese Males, 1980 (using the Formula (7) in II.D.)

$\mathring{e}_t(z)$ for $t \geq z$	Case 1			Case 2			Case 3			Standard values of $\mathring{e}_t$
	Values of z			Values of z			Values of z			
	80	85	90	80	85	90	80	85	90	
$\mathring{e}_{80}(z)$	6.04	----	----	6.05	----	----	5.80	----	----	6.08
$\mathring{e}_{85}(z)$	4.39	4.37	----	4.41	4.37	----	4.11	4.28	----	4.38
$\mathring{e}_{90}(z)$	3.24	3.16	3.15	3.26	3.16	3.15	2.95	3.05	3.13	3.15
$\mathring{e}_{95}(z)$	2.44	2.34	2.30	2.46	2.34	2.29	2.18	2.22	2.26	2.20

and Nanjo (1984) (z=90).

Notices

1. In the iteration of the present computation, the regula falsi was usually used but the bisection method may be used instead when it is hard to meet a given criterion of convergence.

2. The present method should be originally used to obtain the values of $\mathring{e}_z$ (z) for z=80, 85,... , but in Cases 1 and 2, it may be used to get the values of $\mathring{e}_t$ (z) for $t \geq z$. In Case 3, the values of $\mathring{e}_z(z)$ seem to be effectual.

V. Concluding Remarks

1. In this paper, a method was presented to get the values of $\mathring{e}_z$ at old ages using Makeham's curve on the basis of the principle of agreement with the data by Keyfitz.

2. Another approach to the estimation of values of $\mathring{e}_z$ at old ages was given by Horiuchi and Coale (1982). This is based on the assumption of a stable population model.

3. The present method depends upon the precision of formulae (2), (3) and (6). It may not be very difficult to make more precise the formula (5) which provides the values of A_x for $x > z-5$ to be used in the formula (6), if countries and eras concerned are specified.

4. In Case 3 tentative values of population ${}_5K_x$ were used for $x \geqq z$ estimated in each iteration. Improving the method of estimating these populations might also better the present results.

Our method may be more acceptable if this method is used for z= 85, 90 and 95, especially in Case 3.

References

Horiuchi, Shiro and A. J. Coale, 1982. "A simple equation for estimating the expectation of life at old ages," Population Studies, Vol. 36, No. 2, pp.317-326.

Keyfitz, Nathan. 1968. "A life table that agrees with the data: II," Jour. of Amer. Statistical Assoc. 63:1253-1268.

Keyfitz, Nathan and J. Frauenthal. 1975. "An improved life table method," Biometrics 31: 889-899.

Keyfitz, Nathan. 1977. Applied Mathematical Demography. New York: John Wiley.

Kobayashi, Kazumasa and Zenji Nanjo. 1984. "A method of constructing an abridged life table by the combined use of iteration and the cubic spline interpolation," NUPRI Research Paper Series 17, Population Research Institute, Tokyo: Nihon University.

Reed, Lowell J. and Margaret Merrell. 1939. "A short method for constructing an abridged life table," Amer. Jour. Hygiene 30: 33 -62.

Spiegelman, Mortimer. 1968. Introduction to Demography. Cambridge: Harvard University Press.