Reversiblity of a storage process in discrete-time with continuous components

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Abstract

We consider a simple storage process in discrete-time with the continuous components. The process is described by the equation $X_{n+1} = X_n + A_n - B_{n+1}$ where X_n is the system state at time n, A_n and B_{n+1} the infow from outside at the start of n-th time slot and the outflow at the end of n-th time slot, respectively.

We assume that A_n is independent of any other element in the system, and B_{n+1} depends only on the system state at the start of n-th time slot, $X_n + A_n$. Such a storage process is applied to discrete-time queue, fluid queues and so on. For a stationary process of the above system, we investigate the reversibility in time.

Keywords: Quasi-reversibility, discrete-time storage model, time-reversed process.

I Introduction

In this paper, we consider a discrete-time storage process with the continuous components. The model is composed of two kinds of flows, inflow and outflow. At the start of each time slot, the system has the inflow from outside. The amount of inflow is independent of any other element in the system. On the other hand, at the end of time slot there is the outflow whose size depends only on the system state, the system state at that time plus the inflow just after the time slot.

For queues and queueing networks, such models with discrete states were considered by several authors. Walrand (1983) and Ōsawa (1989) studied discrete-time queues with Poisson arrivals and negative binomial arrivals, respectively. They showed their models have quasi-reversibility. Ōsawa (1994) showed that these results hold for queues with the general arrivals. Further, he determined the form of the departure rule under the system has the quasi-reversibility. These results were generalized by Miyazawa (1994), (1995). Time-reversibility and quasi-reversibility have been dealt with precisely in Kelly (1979) and Walrand (1988).

In this paper, we consider the model with the continuous state space. The mathematical model is introduced in section 2, precisely. We define three processes for the system state. In section 3, we deal with the relationship among three processes for the system state. We study the time-reversed properties of the process.

Ⅱ　The model

Consider a discrete-time storage model with two kinds of flows, inflow and outflow. Let A_n be the amount of inflow just after time n and B_{n+1} be the amount of outflow at the end of n th time slot. Then the system state X_n at time n is defined by

$$X_{n+1} = X_n + A_n - B_{n+1}, \quad n \in \boldsymbol{Z},$$

where $\boldsymbol{Z}$ is the set of all integers.

Assume that $\{A_n; n \in \boldsymbol{Z}\}$ are independent and identical random variables which are independent of any other elements, and moreover,

$$\mathrm{P}[A_n \leq y | X_i, B_i, (i \leq n), A_j, (j < n)] = A(y).$$

We also assume that B_{n+1} depends only on $X_n + A_n$, that is,

$$\mathrm{P}[B_{n+1} \leq y | X_i, A_i, B_i, (i \leq n)] = \mathrm{P}[B_{n+1} \leq y | X_n + A_n].$$

The outflows from the system are determined by conditional distribution functions

$$\mathrm{P}[B_{n+1} \leq y | X_n + A_n = x] = B(x, y),$$

for $x, y \geq 0$ and $n = 0, 1, \cdots$. For these distributions, we make assumptions that $A(x)$ and $B(x, y)$ are absolutely continuous on R_+ with probability density functions ($p. d. f.$) $a(y)$ and $b(x, y)$ for each $x \geq 0$, that is,

$$\int_0^\infty a(y)dy = 1, \quad \int_0^\infty b(x,y)dy = 1.$$

Under the above conditions, we consider three stochastic processes:

$$\chi \;= \{(B_n, X_n, A_n) : n \in \boldsymbol{Z}\},$$

$$\chi_A = \{(X_n, A_n) : n \in \boldsymbol{Z}\},$$

$$\chi_B = \{(X_n, B_n) : n \in \boldsymbol{Z}\}.$$

It is clear that these are homogeneous Markov chains. Throughout the paper, these processes are in stationary. In the rest of this paper, we focus on reversibility and relations among these processes.

Ⅲ Reversed properties for processes

In this section, we investigate relations for three processes defined in the previous section. In particular, we are interested in time-reversed properties. For our purpose, assume that the outflow is controlled by

$$b(x, y) = c(x)\alpha(x-y)\beta(y),\ x, y > 0,$$

where $\alpha(x)$, $\beta(x)$ and $c(x)$ are continuous and integrable functions on $\boldsymbol{R}_+$ satisfying that

$$c(x) = \left(\int_0^x \alpha(x-y)\beta(y)dy\right)^{-1} = \{\alpha * \beta(x)\}^{-1}.$$

3.1. Process χ

Let $\boldsymbol{R}_+$ be the set of all real numbers that are positive, then the process χ is a Markov chain with the state space $\boldsymbol{R}_+^3$. For two states $\boldsymbol{x} = (u, x, s)$ and $\boldsymbol{y} = (v, y, t) \in \boldsymbol{R}_+^3$, its transition probability density is given by

$$\begin{aligned} p(\boldsymbol{x}, \boldsymbol{y}) &= b(x+s, v)a(t)1_Q(\boldsymbol{x}, \boldsymbol{y}) \\ &= c(x+s)\alpha(x+s-v)\beta(v)a(t)1_Q(\boldsymbol{x}, \boldsymbol{y}) \end{aligned}$$

where $1_Q(\boldsymbol{x}, \boldsymbol{y})$ is an indicator function for a set $Q = \{(\boldsymbol{x}, \boldsymbol{y}) : x + s = y + v\}$;

$$1_Q(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 1, & (\boldsymbol{x}, \boldsymbol{y}) \in Q, \\ 0, & \text{otherwise}. \end{cases}$$

We now get the following theorem on the stationary distribution for the process χ.

Theorem 1.

Assume that $\beta(y) = \kappa a(y)$ for any positive constant κ, then following statements hold.

(1) The process χ has a stationary distribution given by the probability density function

$$\nu(\boldsymbol{x}) = Ca(u)\alpha(x)a(s),\ \boldsymbol{x} = (u, x, s) \in \boldsymbol{R}_+^3,$$

where C is the normalizing constant.

(2) The process $\chi^- = \{(A_n, X_n, B_n) : n \in \boldsymbol{Z}\}$ is reversed in time for χ.

Proof (1) By direct calculation, we get

$$\int_{R_+^3} v(\boldsymbol{x})p(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}$$
$$= \int_{R_+^3} Ca(u)\alpha(x)a(s)c(x+s)\alpha(x+s-v)\kappa a(v)a(t)1_Q(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}$$
$$= \int_0^{\infty} a(u)du\left(c(y+v)\int_0^{y+v}\alpha(x)\kappa a(y+v-x)dx\right)C\alpha(y)a(v)a(t)$$
$$= v(\boldsymbol{y})$$

for any $\boldsymbol{y} \in \boldsymbol{R}_+^3$. This means the stationary distribution for the process χ is given by $\{v(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{R}_+^3\}$.

(2) For two processes χ and χ^-, denote system states at time n by $\boldsymbol{X}_n = (B_n, X_n, A_n)$ and $\boldsymbol{X}_n^- = (A_n, X_n, B_n)$ respectively. Moreover, define a state $\boldsymbol{x}^- = (s, x, u)$ for $\boldsymbol{x} = (u, x, s)$, we then have

$$v(\boldsymbol{x})p(\boldsymbol{x},\boldsymbol{y}) = Ca(u)\alpha(x)a(s)c(x+s)\alpha(x+s-v)\kappa a(v)a(t)1_Q(\boldsymbol{x},\boldsymbol{y})$$
$$= Ca(v)\alpha(y)a(t)c(y+v)\alpha(y+v-x)\kappa a(s)a(u)1_Q(\boldsymbol{x},\boldsymbol{y})$$
$$= v(\boldsymbol{y}^-)p(\boldsymbol{y}^-,\boldsymbol{x}^-)$$

for $\boldsymbol{x} = (u, x, s)$ and $\boldsymbol{y} = (v, y, t) \in \boldsymbol{R}_+^3$. This means that the joint events $(\boldsymbol{X}_n, \boldsymbol{X}_{n+1})$ and $(\boldsymbol{X}_{n+1}^-, \boldsymbol{X}_n^-)$ have the same probabilistic law. Thus the second statement of Theorem holds.

Remark

The property shown in (2) of Theorem 1 is known as quasi-reversibility.

3.2. Processes χ_A and χ_B

By virtue of Theorem 1, we can have the relations between processes χ_A and χ_B.

The process χ_A is a Markov chain with transition probabilities given by

$$p_A(\boldsymbol{x},\boldsymbol{y}) = b(x+s, x+s-y)a(t),\ \boldsymbol{x} = (x, s),\ \boldsymbol{y} = (y, t) \in \boldsymbol{R}_+^2.$$

In the same way, the process χ_B is a Markov chain with transition probabilities given by

$$p_B(\boldsymbol{x},\boldsymbol{y}) = a(y+v-x)b(y+v, v),\ \boldsymbol{x} = (x, u),\ \boldsymbol{y} = (y, v) \in \boldsymbol{R}_+^2.$$

Theorem 2.

Under the same condition of Theorem 1, processes χ_A and χ_B are time-reversed each other.

Proof　From Theorem 1, χ_A and χ_B have the same stationary distribution given by $\pi(\boldsymbol{x}) = C\alpha(x)a(s)$ for $\boldsymbol{x} = (x, s)$. We now have

$$\pi(\boldsymbol{x})\,p_A(\boldsymbol{x},\boldsymbol{y}) = C\alpha(x)a(s)c(x+s)\alpha(y)\kappa a(x+s-y)a(t)$$
$$= C\alpha(y)a(t)a(x+s-y)c(x+s)\alpha(x)\kappa a(s)$$
$$= \pi(\boldsymbol{y})p_B(\boldsymbol{y},\boldsymbol{x})$$

for $\boldsymbol{x}$ = (x, s) and $\boldsymbol{y}$ = $(y, t) \in \boldsymbol{R}_+^2$. Thus this theorem has been proved.

3.3. Process for the system state

Consider the process $\{X_n : n \in \boldsymbol{Z}\}$ that describes the system state. This process is a Markov chain with transition probabilities given by

$$p(x,y) = \begin{cases} \int_{y-x}^{\infty} b(x+s, x+s-y)a(s)ds, & y > x, \\ \int_{0}^{\infty} b(x+s, x+s-y)a(s)ds, & y \leq x, \end{cases}$$

for $x, y \in \boldsymbol{R}_+$. Then we obtain the following theorem.

Theorem 3.

Under the same condition of Theorem 1, the process $\{X_n\}$ is time-reversible.

Proof From Theorem 1, the process $\{X_n\}$ has the stationary distribution given by $\pi(x) = C\alpha(x)$ for $x \in \boldsymbol{R}_+$. For $y > x$, we then have

$$\begin{aligned} \pi(x)p(x,y) &= C\alpha(x)\int_{y-x}^{\infty} c(x+s)\alpha(y)\kappa a(x+s-y)a(s)ds \\ &= C\alpha(y)\int_{0}^{\infty} c(y+t)\alpha(x)\kappa a(y+t-x)a(t)dt \\ &= \pi(y)p(y,x) \end{aligned}$$

for $y > x > 0$. This means that $\{X_n\}$ is time-reversible.

Ⅳ An example for queues

Consider the GI/G/1 queueing storage model in discrete-time. At time n, a customer arrives at the system and he/she needs the service time A_n whose size has a *p.d.f.* $f(x)$. Let B_{n+1} be the amount of work in the system during times n and $n+1$ and suppose that B_{n+1} has a *p.d.f.* $b(x, y)$ under the condition that the total work road is x at the beginning of n-th time slot. Then the system state X_n is defined by

$$X_{n+1} = X_n + A_n - B_{n+1},\ n \in \boldsymbol{Z}.$$

If $b(x, y) = c(x)\alpha(x-y)f(y)$, then the process $\{X_n : n \in \boldsymbol{Z}\}$ is time-reversible. From quasi-reversibility in Theorem 1, the inflow process $\{A_n\}$ and the outflow process $\{B_n\}$ are independent each other and have the same probabilistic movements.

For example, in the case of $f(x) = \mu e^{-\mu x}$, i.e., M/G/1 storage model, the system has quasi-reversibility for any function $\alpha(x)$. This means that we can select the outflow rule to control the system state.

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