Note

A Note on the Stock Market Trend Analysis Using
Markov-Switching EGARCH Models

MITSUI, Hidetoshi*

1 Introduction

The stock market's fluctuations follow a continuous trend whether or not is still a topic with mixed opinions. However, if they do, it may be possible to discern fluctuation trends, that is, identify bull markets and bear markets, using a statistical model. Many methods have been applied to this problem. In particular, there are many prior studies based on the Markov-switching model, including Maheu and McCurdy (2000). Maheu and McCurdy (2000) suggested identifying trend changes by switching means of time-series model according to Markov process state variables. This model uses an extension of the Markov-switching ARCH model, suggested by Hamilton and Susmel (1994) and Cai (1994), for volatility variation. The Markov-switching model is frequently used to formularize volatility. Other representative works include the Markov-switching GARCH model by Gray (1996), Klaassen (2002) and Haas *et al.* (2004), and the Markov-switching stochastic volatility model by So *et al.* (1998).

Trend analysis on asset prices using Markov-switching models normally classify trends into two regimes – bear markets and bull markets. The mean of the rate of variation in stock prices is either a negative or positive value. When the value continues to be positive there is an upward trend (bull market), and when the value continues to be negative there is a downward trend (bear market). The assumption is that the two states follow a Markov process. However, Maheu *et al.* (2009) stated that investors distinguish more detailed trends including temporary trends, and suggested a 4-state Markov-switching model identifying a bear negative, bear positive, bull negative and bull positive growth. Therefore, this study suggests a 4-state Markov-switching EGARCH model, which combines the 4-state Markov-switching model and an Exponential GARCH (EGARCH) model.

The reason why an EGARCH model is used for formularizing volatility is because there is an asymmetry in stock market volatility. That is, volatility is higher the day after stock prices fall compared to the day after stock prices rise. Furthermore, because volatility involves very high and very low states that cannot be expressed with an EGARCH model, the model switches volatility variations also. However, because it is not always the case that volatility switching occurs at the same time as

* Associate Professor, College of Economics, Nihon University, E-mail: mitsui.hidetoshi@nihon-u.ac.jp

trends change (from bull to bear, or from bear to bull), the model accommodates expected rate of return and volatility switching separately, with two state variables that follow a Markov process. In this note, this model shall be called a Markov-switching EGARCH (MSEGARCH) model.

Satoyoshi and Mitsui (2011) used the MS-EGARCH model to analyze the Nikkei 225 options market and showed that the MS-EGARCH model is effective for identifying bull and bear markets of the Nikkei Stock Average, which are underlying assets. Therefore, this study extends the MSEGARCH model to analyze bull and bear markets with more detail. First, the asymmetry of volatility is studied to see if it differs in bull and bear markets. Next, we see whether the transition probability of bull and bear markets are influenced by the rate of return on the previous term, in other words, whether a price growth in a bull market increases the probability of a continued bull market, and whether a price fall in a bear market increases the probability of a continued bear market. The transition probabilities of state variables are considered time-varying transition probabilities. According to the above, this study suggests a model that allows trend analysis using a Markov-switching model.

The brief descriptions of the following chapters are as follows: Chapter 2 describes the 4-state Markov-switching model, the 4-state Markov-switching EGARCH model, the asymmetry model and the time-varying transition probability model. Chapter 3 explains the trend analysis model. Chapter 4 contains the conclusion.

2 Model

2.1 4-state Markov-Switching Model

In this chapter, the author will explain briefly the 4-state Markov-switching model by Maheu *et al.* (2009). When R_t is the rate of return on asset price at time t, the process of R_t can be expressed as follows:

$$R_t = \mu + \sigma z_t, z_t \sim i.i.d., E[z_t] = 0, V[z_t] = 1, \tag{2.1}$$

$$\mu = \mu_1 \Delta_{1t} + \mu_2 \Delta_{2t} + \mu_3 \Delta_{3t} + \mu_4 \Delta_{4t}, \tag{2.2}$$

$$\sigma = \sigma_1 \Delta_{1t} + \sigma_2 \Delta_{2t} + \sigma_3 \Delta_{3t} + \sigma_4 \Delta_{4t}, \tag{2.3}$$

$$p_{ij} = \Pr[\Delta_t = j \mid \Delta_{t-1} = i], i,j = 1, 2, 3, 4. \tag{2.4}$$

Here, the constant term μ is the mean, z_t is the error term, and the assumption is that there is no autocorrelation in returns. *i.i.d.* expresses an independently and identically distributed. Δ_t is a state variable that follows a first-order Markov process. $(\Delta_{1t}, \Delta_{2t}, \Delta_{3t}, \Delta_{4t})$ is $(1, 0, 0, 0)$ when $\Delta_t = 1$, $(0, 1, 0, 0)$ when $\Delta_t = 2$, $(0, 0, 1, 0)$ when $\Delta_t = 3$ and $(0, 0, 0, 1)$ when $\Delta_t = 4$. Also, the assumption is that mean μ and volatility σ follow Δ_t and switch simultaneously. The following restrictions apply to μ.

$\mu_1 < 0 \cdots$ bear negative growth,

$\mu_2 > 0 \cdots$ bear positive growth,

$\mu_3 < 0 \cdots$ bull negative growth,

$\mu_4 > 0 \cdots$ bull positive growth.

A bear positive growth is a temporary rise during a long-term downward trend. A bull negative growth is a temporary fall during a long-term upward trend. The transition matrix of Δ_t is:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{21} & 0 & p_{41} \\ p_{12} & p_{22} & 0 & 0 \\ 0 & 0 & p_{33} & p_{43} \\ p_{14} & 0 & p_{34} & p_{44} \end{pmatrix}. \tag{2.5}$$

Here, p_{14} is the probability of a transition from bear to bull, and p_{41} is the probability of a transition from bull to bear. The path from bear to bull and the path from bull to bear are each one path, making it easier to identify between states for prediction. This model can be called the 4-state （1） model.

The model below is when mean μ and volatility σ transition independently. There are two volatility states to avoid increasing the number of parameters.

$$R_t = \mu + \sigma z_t, z_t \sim i.i.d., E[z_t] = 0, V[z_t] = 1, \tag{2.6}$$

$$\mu = \mu_1 \Delta_{1t} + \mu_2 \Delta_{2t} + \mu_3 \Delta_{3t} + \mu_4 \Delta_{4t}, \tag{2.7}$$

$$\sigma = \sigma_1 \Gamma_{1t} + \sigma_2 \Gamma_{2t}, \sigma_1 < \sigma_2, \tag{2.8}$$

$$p_{ij} = \Pr[\Delta_t = j \mid \Delta_{t-1} = i], i,j = 1, 2, 3, 4, \tag{2.9}$$

$$q_{kl} = \Pr[\Gamma_t = l \mid \Gamma_{t-1} = k], k,l = 1, 2. \tag{2.10}$$

Here, Γ_t is the state variable that follows a Markov-chain, and it is independent. $(\Gamma_{1t}, \Gamma_{2t})$ is $(1, 0)$ when $\Gamma_t = 1$ and $(0, 1)$ when $\Gamma_t = 2$. The transition matrix of Δ_t and Γ_t are each:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{21} & 0 & p_{41} \\ p_{12} & p_{22} & 0 & 0 \\ 0 & 0 & p_{33} & p_{43} \\ p_{14} & 0 & p_{34} & p_{44} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix}. \tag{2.11}$$

This model can be called the 4-state（2） model.

With only a bull and bear growth, the μ, p_{ij} and $\mathbf{P}$ of the 4-state （1） and 4-state（2） models are as follows:

$$\mu = \mu_1 \Delta_{1t} + \mu_2 \Delta_{2t}, \mu_1 < \mu_2, \tag{2.12}$$

$$p_{ij} = \Pr[\Delta_t = j \mid \Delta_{t-1} = i], i,j = 1, 2, \quad \mathbf{P} = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix}. \tag{2.13}$$

As in Eq. (2.12), the restriction of $\mu_1 < \mu_2$ will be applied. Depending on the asset price the

estimated results may or may not be $\mu_1 < 0$, $\mu_2 > 0$, but we shall call μ_1 a bear growth and μ_2 a bull growth. These models can be called the 2-state（1）and 2-state（2）models.

2.2   4-state Markov-Switching EGARCH Model

Next is the 4-state Markov-switching EGARCH model, an extension of the 4-state Markov-switching model of section 2.1.

$$R_t = \mu_a + \phi_a R_{t-1} + \sqrt{V_{ab,t}} z_t,\ z_t \sim i.i.d.,\ E[z_t] = 0,\ V[z_t] = 1, \tag{2.14}$$

$$\ln(V_{ab,t}) = \omega_b + \beta \ln(V_{ab,t-1}) + \theta\left[\frac{R_{t-1} - \mu_a - \phi_a R_{t-2}}{\sqrt{V_{ab,t-1}}}\right]$$

$$+ \gamma\left[\left|\frac{R_{t-1} - \mu_a - \phi_a R_{t-2}}{\sqrt{V_{ab,t-1}}}\right| - E[|z_{t-1}|]\right], \tag{2.15}$$

$$\mu_a = \mu_1 \Delta_{1t} + \mu_2 \Delta_{2t} + \mu_3 \Delta_{3t} + \mu_4 \Delta_{4t}, \tag{2.16}$$

$$\phi_a = \phi_1(\Delta_{1t} + \Delta_{2t}) + \phi_2(\Delta_{3t} + \Delta_{4t}), \tag{2.17}$$

$$\omega_b = \omega_1 \Gamma_{1t} + \omega_2 \Gamma_{2t},\ \omega_1 < \omega_2, \tag{2.18}$$

$$p_{ij} = \Pr[\Delta_t = j \mid \Delta_{t-1} = i],\ i,j = 1, 2, 3, 4, \tag{2.19}$$

$$q_{kl} = \Pr[\Gamma_t = l \mid \Gamma_{t-1} = k],\ k,l = 1, 2. \tag{2.20}$$

Here, coefficient ϕ_a of R_{t-1} is ϕ_1 in a bear negative or bear positive growth, and ϕ_2 in a bull negative or bull positive growth. $V_{ab,t}$ expresses the conditional variance of R_t as $V_{ab,t} = V[R_t \mid \Delta_t = a,\ \Gamma_t = b,\ I_{t-1}]$, with the condition of information $I_{t-1} = \{R_{t-1}, R_{t-2}, \cdots\}$ until the term $\Delta_t = a$, $\Gamma_t = b$, $t-1$. Δ_t and Γ_t are state variables that follow a Markov-chain, and they are independent. The transition matrix of Δ_t and Γ_t is as follows:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{21} & 0 & p_{41} \\ p_{12} & p_{22} & 0 & 0 \\ 0 & 0 & p_{33} & p_{43} \\ p_{14} & 0 & p_{34} & p_{44} \end{pmatrix},\quad \mathbf{Q} = \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix}. \tag{2.21}$$

It is known that estimating parameters becomes difficult when a Markov-switching model and volatility variation model are simply combined, due to the dependence of state variables. However, this study's model makes estimation possible with maximum-likelihood estimation by using the idea from the Markov-switching GARCH model by Haas *et al.*（2004）. This model can be called the 4-state（2）-MSEG model.

With only bull and bear growth, the transition matrix $\mathbf{P}$ of μ_a, ϕ_a, p_{ij}, Δ_t of the above 4-state（2）-MSEG Model is as follows:

$$\mu_a = \mu_1 \Delta_{1t} + \mu_2 \Delta_{2t}, \ \mu_1 < \mu_2 \tag{2.22}$$

$$\phi_a = \phi_1 \Delta_{1t} + \phi_2 \Delta_{2t}, \tag{2.23}$$

$$p_{ij} = \Pr[\Delta_t = j \mid \Delta_{t-1} = i], \ i,j = 1, 2, \ \mathbf{P} = \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix}. \tag{2.24}$$

This model can called the 2-state（2）-MSEG Model.

2.3　Asymmetry Model

The present study focuses on the asymmetry model, whereby the asymmetry of volatility in the bullish sentiment differs from that in the bear sentiment. Here, the Eq. (2.15) is extended into the following model, whereby the parameter θ, which captures the asymmetry of volatility, can also go through switching.

$$\ln(V_{ab,t}) = \omega_b + \beta \ln(V_{ab,t-1}) + \theta_a \left[\frac{R_{t-1} - \mu_a - \phi_a R_{t-2}}{\sqrt{V_{ab,t-1}}} \right]$$

$$+ \gamma \left[\left| \frac{R_{t-1} - \mu_a - \phi_a R_{t-2}}{\sqrt{V_{ab,t-1}}} \right| - E[|z_{t-1}|] \right], \tag{2.25}$$

$$\theta_a = \theta_1 \Delta_{1t} + \theta_2 \Delta_{2t}. \tag{2.26}$$

This model can be called the "as" model for convenience.

2.4　Time-varying Transition Probability Model

This note also focuses on the time-varying transition probability model, whereby the probability of transition between the bull and bear markets sentiment is influenced by the return of the previous period.

$$p_{11,t} = \Pr[\Delta_t = 1 \mid \Delta_{t-1} = 1, R_{t-1}] = \frac{\exp(\lambda_0 + \lambda_1 R_{t-1})}{1 + \exp(\lambda_0 + \lambda_1 R_{t-1})}, \tag{2.27}$$

$$p_{22,t} = \Pr[\Delta_t = 2 \mid \Delta_{t-1} = 2, R_{t-1}] = \frac{\exp(\xi_0 + \xi_1 R_{t-1})}{1 + \exp(\xi_0 + \xi_1 R_{t-1})}, \tag{2.28}$$

Here, if $\lambda_1 < 0$, p_{11} becomes higher when $R_{t-1} < 0$. That is, it becomes highly probable that a decrease in stock prices will maintain the bear market sentiment. In case of $\xi_1 > 0$, p_{22} becomes higher when $R_{t-1} > 0$. Then, $p_{12,t}$ and $p_{43,t}$ can be expressed as follows:

$$p_{12,t} = \Pr[\Delta_t = 2 \mid \Delta_{t-1} = 1, R_{t-1}] = \frac{\exp(\zeta)}{1 + \exp(\lambda_0 + \lambda_1 R_{t-1}) + \exp(\zeta)}, \tag{2.29}$$

$$p_{43,t} = \Pr\left[\Delta_t = 3 \mid \Delta_{t-1} = 4, R_{t-1}\right] = \frac{\exp(v)}{1 + \exp(\xi_0 + \xi_1 R_{t-1}) + \exp(v)} , \tag{2.30}$$

Thus, it becomes highly probable that a rise in stock prices will maintain the bull market sentiment. Hence, this model can be called the "tv" model for convenience.

3　Trend Analysis Model

When conducting a stock market trend analysis, if in the 4-state(2)-MSEG-tv the error term follows t distribution, the following model can be derived from Eqs. (2.14) – (2.21) and Eqs. (2.27) – (2.30) :

$$R_t = \mu_a + \phi_a R_{t-1} + \sqrt{V_{ab,t}}\, z_t, \ z_t \sim i.i.d.t\,(0, 1, v)$$

$$\ln(V_{ab,t}) = \omega_b + \beta \ln(V_{ab,t-1}) + \theta\left[\frac{R_{t-1} - \mu_a - \phi_a R_{t-2}}{\sqrt{V_{ab,t-1}}}\right]$$

$$+ \gamma\left[\left|\frac{R_{t-1} - \mu_a - \phi_a R_{t-2}}{\sqrt{V_{ab,t-1}}}\right| - E\left[\,|z_{t-1}|\,\right]\right],$$

$$\mu_a = \mu_1 \Delta_{1t} + \mu_2 \Delta_{2t} + \mu_3 \Delta_{3t} + \mu_4 \Delta_{4t},$$

$$\phi_a = \phi_1\,(\Delta_{1t} + \Delta_{2t}) + \phi_2\,(\Delta_{3t} + \Delta_{4t}),$$

$$\omega_b = \omega_1 \Gamma_{1t} + \omega_2 \Gamma_{2t}, \ \omega_1 < \omega_2,$$

$$p_{ij} = \Pr[\Delta_t = j \mid \Delta_{t-1} = i], \ i,j = 1, 2, 3, 4,$$

$$q_{kl} = \Pr[\Gamma_t = l \mid \Gamma_{t-1} = k], \ k,l = 1, 2.$$

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{21} & 0 & p_{41} \\ p_{12} & p_{22} & 0 & 0 \\ 0 & 0 & p_{33} & p_{43} \\ p_{14} & 0 & p_{34} & p_{44} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix}.$$

$$p_{11,t} = \Pr[\Delta_t = 1 \mid \Delta_{t-1} = 1, R_{t-1}] = \frac{\exp(\lambda_0 + \lambda_1 R_{t-1})}{1 + \exp(\lambda_0 + \lambda_1 R_{t-1})} ,$$

$$p_{22,t} = \Pr[\Delta_t = 2 \mid \Delta_{t-1} = 2, R_{t-1}] = \frac{\exp(\xi_0 + \xi_1 R_{t-1})}{1 + \exp(\xi_0 + \xi_1 R_{t-1})} ,$$

$$p_{12,t} = \Pr\left[\Delta_t = 2 \mid \Delta_{t-1} = 1, R_{t-1}\right] = \frac{\exp(\zeta)}{1 + \exp(\lambda_0 + \lambda_1 R_{t-1}) + \exp(\zeta)} ,$$

$$p_{43,t} = \Pr\left[\Delta_t = 3 \mid \Delta_{t-1} = 4, R_{t-1}\right] = \frac{\exp(v)}{1 + \exp(\xi_0 + \xi_1 R_{t-1}) + \exp(v)} ,$$

Therefore, the parameter $\{\mu_1,\ \mu_2,\ \mu_3,\ \mu_4,\ \phi_1,\ \phi_2,\ \omega_1,\ \omega_2,\ \beta,\ \theta,\ \gamma,\ v, p_{22}, p_{33}, q_{11}, q_{22},\ \lambda_0,\ \lambda_1,\ \xi_0,\ \xi_1,\ \zeta,\ v\}$ can be estimated using maximum likelihood estimation[1] . Moreover, a variety of combinations

---

[1]　For details, refer to Mitsui (2011)，Mitsui (2012).

can be used using the 4-state（2）-MSEG-as model, 2-state（2）-MSEG-tv model, 2-state（2）-MSEG-as model, etc.

4　Summary

This note suggests a 4-state Markov-switching EGARCH model, which is a combination of a 4-state Markov-switching model and EGARCH model, and explains the use of an asymmetry model and time-varying transition probability model in order to conduct trend analysis in more detail. These models can be used to conduct empirical analysis on actual stock index data such as the Nikkei Stock Average, TOPIX, etc.

References

[1]　Cai, J.（1994）, "A Markov Model of Switching-Regime ARCH," *Journal of Business & Economic Statistics*, 12, pp. 309–316.

[2]　Gray, S. F.（1996）, "Modeling the Conditional Distribution of Interest Rates as a Regime-Switching Process," *Journal of Financial Economics*, 42, pp. 27–62.

[3]　Haas, M., Mittnik, S. and Paolella, M. S.（2004）, "A New Approach to Markov-Switching GARCH Models," *Journal of Financial Econometrics*, 2, pp. 493–530.

[4]　Hamilton, J. D. & Susmel, R.（1994）, "Autoregressive Conditional Heteroskedasticity and Changes in Regime," *Journal of Econometrics*, 64, pp. 307–333.

[5]　Henry, O. T.（2009）, "Regime Switching in the Relationship between Equity Returns and Short-Term Interest Rates in the UK," *The Journal of Banking & Finance*, 33, pp. 405–414.

[6]　Klaassen, F.（2002）, "Improving GARCH Volatility Forecasts with Regime-Switching GARCH," *Empirical Economics*, 27, pp. 363–394.

[7]　Lunde, A. and Timmermann, A.（2004）, "Duration Dependence in Stock Prices: An Analysis of Bull and Bear Markets," *Journal of Business & Economic Statistics*, 22, pp. 253–273.

[8]　Maheu, J. M. and McCurdy, T. H.（2000）, "Identifying Bull and Bear Markets in Stock Returns," *Journal of Business & Economic Statistics*, 18, pp. 100–112.

[9]　Maheu, J. M., McCurdy, T. H. and Song, Y.（2009）, "Extracting Bull and Bear Markets from Stock Returns," Working Paper from University of Toronto, Department of Economics.

[10]　Mitsui, H.（2011）, "A Note on the 4-state Markov-Switching EGARCH Model," *KEIZAI SHUSHI*（The Nihon University Economic Review）, Vol.81,　No.3,　pp. 165–169.（in Japanese）.

[11]　――――（2012）, "Application of Markov-Switching EGARCH Models to the Nikkei Stock Average Variation," *KEIZAI SHUSHI*（The Nihon University Economic Review）, Vol.82,　No.1,　pp. 79–84.

[12]　Satoyoshi, K. and Mitsui, H.（2011）, An Empirical Study of Option Pricing under Bull and Bear Market Conditions," *Journal of Business Research*, Institute of Business Research, College of Economics, Nihon University, No.33, pp. 63–87.（in Japanese）.

[13]　So, M. K. P., Lam, K. & Li, W. K.（1998）, "A Stochastic Volatility Model with Markov Switching," *Journal of Business & Economic Statistics*, 16, pp. 244–253.