

A Note on the Option Pricing Using a Normal Mixture EGARCH Model

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1. Introduction

This note will explain an option evaluation method using a normal mixture EGARCH model. In a stock market, the volatility of stock prices fluctuates significantly depending on the condition of the market, with the distribution of rates of return known for having thicker tails and being less symmetrical than a normal distribution. For stock price indices, the volatility tends to be higher the day after a drop in price than it is the day after an increase in price. This phenomenon is known as asymmetric volatility or the leverage effect. Some existing research on option price evaluation has looked at option evaluation including the leverage effect, but there is only limited research considering the distortion of the distribution.

Empirical studies on option pricing using an ARCH model include Engle and Mustafa (1992), Noh *et al.* (1994), Saez (1997), Sabbatini and Linton (1998), Bauwens and Lubrano (1998), Duan and Zhang (2001), Bauwens and Lubrano (2002), and Christoffersen and Jacobs (2004). In Japanese option market, Nikkei 225 options were looked at by Moriyasu (1999), Mitsui (2000), Mitsui and Watanabe (2003), Watanabe (2003), Takeuchi-Nogimori and Watanabe (2008), Satoyoshi and Mitsui (2011), and Satoyoshi and Mitsui (2012). Moreover, Takeuchi-Nogimori (2012) and Ubukata and Watanabe (2013) used high frequency data with a realized ARCH model for analysis. Several of these existing studies evaluated option prices including the leverage effect, but none considered distortion of the distribution.

Haas *et al.* (2004) and Alexander and Lazar (2006) proposed the normal mixture GARCH model combining the GARCH model with normal mixture distributions to deal with the thick tails and asymmetry of the rate of return distribution. By having different values for the average of each normal distribution that the normal mixture distribution is comprised of, the overall distribution becomes one with distorted asymmetry. The different variance values allow the thickness of the tails to be expressed, and with the relative simplicity of the probability density function it has been used repeatedly for financial empirical research. In this note, the normal mixture GARCH model of Haas *et al.* (2004) and Alexander and Lazar (2006) will be used as a base along with turning fluctuations in volatility into an EGARCH model to propose a mixture normal EGARCH model and apply it to the evaluation of option prices. European option prices like Nikkei 225 options can be easily calculated

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with a Monte Carlo simulation assuming risk neutrality of the investor.

The brief descriptions of the following chapters are as follows: Chapter 2 describes the normal mixture EGARCH model, investor risk neutrality and estimation method. Chapter 3 explains the method for evaluating European options using the Monte Carlo simulation. Chapter 4 contains conclusion.

2. Model

2.1 Normal Mixture EGARCH Model

When random variable X is following a normal mixture distribution composed of K normal distributions, the probability density function of X is

$$f(x) = \sum_{i=1}^K p_i \phi(x; \mu_i, \sigma_i^2). \quad (2.1)$$

Here, weighted value p_i is $p_i > 0$, $\sum_{i=1}^K p_i = 1$, and each component $\phi(x; \mu_i, \sigma_i^2)$ is

$$\phi(x; \mu_i, \sigma_i^2) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-\frac{(x - \mu_i)^2}{2\sigma_i^2}\right]. \quad (2.2)$$

In this note, we express this as

$$X \sim \text{NM}(p_1, \dots, p_K; \mu_1, \dots, \mu_K; \sigma_1^2, \dots, \sigma_K^2). \quad (2.3)$$

Here, NM is an abbreviation for “normal mixture” .

In this note, we assume that the volatility of each normal distribution composing the normal mixture distribution will fluctuate according to the EGARCH model, in order to express the leverage effect. This model is an extension of the normal mixture GARCH model of Haas *et al.* (2004) and Alexander and Lazar (2006), and will be referred to as the normal mixture EGARCH model below. With R_t as the rate of return on the cost of the underlying asset, the normal mixture EGARCH model is expressed as

$$R_t | I_{t-1} \sim \text{NM}(p_1, \dots, p_K; \mu_1, \dots, \mu_K; \sigma_{1t}^2, \dots, \sigma_{Kt}^2), \quad (2.4)$$

$$\ln \sigma_{it}^2 = \omega_i + \beta_i \ln \sigma_{i,t-1}^2 + \theta z_{i,t-1} + \gamma (|z_{i,t-1}| - E(|z_{i,t-1}|)), \quad (2.5)$$

$$z_{i,t-1} = [R_{t-1} - E(R_{t-1} | I_{t-2})] / \sigma_{i,t-1}, \quad i = 1, 2, \dots, K. \quad (2.6)$$

I_{t-1} is an information set up to the time $t-1$, and is $I_{t-1} = \{R_{t-1}, R_{t-2}, \dots\}$. The $E(|z_{i,t-1}|)$ in Eq. (2.5) is $\sqrt{2/\pi}$. In Eqs. (2.5) and (2.6), the volatility σ_{it}^2 value of the component i at time t is dependent on

the $\sigma_{i,t-1}^2$ of the same i component one phase previous and of prediction error $R_{t-1} - E(R_{t-1} | I_{t-2})$ standardized $z_{i,t-1}$ by $\sigma_{i,t-1}$.

In this note, two values are taken at each time t for the average μ_i and the volatility σ_{it}^2 . Investors are also assumed to be risk neutral. At this time, the expected rate of return is equivalent to the rate of return on a risk-free asset, and the weighted values, or in other words the probability of each normal distribution, are the values fulfilling these conditions. One of two values will be taken at each time for volatility σ_{it}^2 as well, but there is no need for the probability of that value to meet the assumption of risk neutrality. Now, the number of normal distributions composing the normal mixture distribution is four instead of two, or in other words $K = 4$ for Eq. (2.4). Next, μ_i , ω_i , β_i will have the following constraints

$$\mu_1 = \mu_2, \quad \mu_3 = \mu_4, \quad \omega_1 = \omega_3, \quad \omega_2 = \omega_4, \quad \beta_1 = \beta_3, \quad \beta_2 = \beta_4. \quad (2.7)$$

Based on these constraints on ω_i and β_i , σ_{it}^2 will be

$$\sigma_{1t}^2 = \sigma_{3t}^2, \quad \sigma_{2t}^2 = \sigma_{4t}^2. \quad (2.8)$$

By redefining these parameters here as $\mu_a \equiv \mu_1, \mu_2$, $\mu_b \equiv \mu_3, \mu_4$, $\omega_a \equiv \omega_1, \omega_3$, $\omega_b \equiv \omega_2, \omega_4$, $\beta_a \equiv \beta_1, \beta_3$, $\beta_b \equiv \beta_2, \beta_4$, $\sigma_{at}^2 \equiv \sigma_{1t}^2, \sigma_{3t}^2$, $\sigma_{bt}^2 \equiv \sigma_{2t}^2, \sigma_{4t}^2$, the average and volatility will be determined. With a probability of $p_1 + p_2$ the average will be μ_a , with a probability of $p_3 + p_4$ it will be μ_b , and volatility will be σ_{at}^2 with a probability of $p_1 + p_3$ and σ_{bt}^2 with a probability of $p_2 + p_4$. Therefore, even with the constraint of the expected rate of return being the same as a risk-free asset due to risk neutrality the volatility value is determined regardless. The expected rate of return and variance are

$$E(R_t | I_{t-1}) = (p_1 + p_2)\mu_a + (p_3 + p_4)\mu_b, \quad (2.9)$$

$$V(R_t | I_{t-1}) = (p_1 + p_3)\sigma_{at}^2 + (p_2 + p_4)\sigma_{bt}^2 + (p_1 + p_2)\mu_a^2 + (p_3 + p_4)\mu_b^2 - \{(p_1 + p_2)\mu_a + (p_3 + p_4)\mu_b\}^2. \quad (2.10)$$

When dealing with the thickness of the tails of the distribution, a t distribution with thicker tails than a normal distribution may be used. Considering the possibility that a normal mixture distribution will not be enough to handle the thickness of the tails, we will also propose a t mixture distribution with t distributions as components. With ν as the degree of freedom, a t mixture distribution with four components is expressed as

$$R_t | I_{t-1} \sim tM(p_1, \dots, p_4; \mu_1, \dots, \mu_4; \sigma_{1t}^2, \dots, \sigma_{4t}^2; \nu). \quad (2.11)$$

tM is an abbreviation of “ t mixture”. Each t distribution composing the t mixture distribution is standardized so that the volatility of each t distribution is σ_{it}^2 ($i = 1, \dots, 4$). The volatility of each t distribution follows the EGARCH model, so this model will be called the t mixture-EGARCH model. The $E(|z_{t-1}|)$ of Eq. (2.5) $[2\sqrt{v-2}\Gamma((v+1)/2)] / [(v-1)\Gamma(v/2)\sqrt{\pi}]$.

2.2 Investor Risk Neutrality and Estimation Method

S_t will be the underlying asset value of options at time t . For this study, the rate of return R_t for underlying asset value at time t can be defined as

$$R_t = \frac{S_t - S_{t-1}}{S_{t-1}}. \quad (2.12)$$

Investors will be assumed to be risk neutral. In a risk neutral world, investors would not demand risk premiums, and so with the interest rate of risk-free assets at time t set as r_t , then the expected rate of return $E(R_t | I_{t-1})$ on underlying assets at time t with the information I_{t-1} available up to time $t-1$ would be equivalent to r_t . Therefore, with risk neutrality and Eq. (2.9), the following equation is true:

$$(p_1 + p_2)\mu_a + (p_3 + p_4)\mu_b = r_t. \quad (2.13)$$

This equation can be rearranged to

$$\mu_{bt} = \frac{r_t - (p_1 + p_2)\mu_a}{p_3 + p_4}. \quad (2.14)$$

However, considering the fluctuation of the interest rate r_t during the observation period, μ_b has been replaced with μ_{bt} . Therefore in this note, in order to fulfill the assumption of investor risk neutrality the constraint of Eq. (2.14) has been placed on the average for the normal mixture distribution. With this, Eq. (2.6) becomes

$$z_{i,t-1} = (R_{t-1} - r_{t-1}) / \sigma_{i,t-1}, \quad i = a, b. \quad (2.15)$$

The maximum likelihood method can be used to estimate model parameters. With L being the likelihood function, the likelihood function for the normal mixture EGARCH model would be

$$L = \sum_{t=1}^T [p_1 \phi(R_t; \mu_a, \sigma_{at}^2) + p_2 \phi(R_t; \mu_a, \sigma_{bt}^2) + p_3 \phi(R_t; \mu_{bt}, \sigma_{at}^2) + p_4 \phi(R_t; \mu_{bt}, \sigma_{bt}^2)]. \quad (2.16)$$

Here,

$$\phi(R_t; \mu_i, \sigma_{jt}^2) = \frac{1}{\sqrt{2\pi\sigma_{jt}^2}} \exp \left[-\frac{(R_t - \mu_i)^2}{2\sigma_{jt}^2} \right], \quad i = a, bt, \quad j = a, b. \quad (2.17)$$

In the case of the t mixture-EGARCH model, it is

$$\phi(R_t; \mu_i, \sigma_{jt}^2) = \frac{\Gamma[(v+1)/2]}{\pi^{1/2} \Gamma(v/2)} \left[1 + \frac{(R_t - \mu_i)^2}{\sigma_{jt}^2 (v-2)} \right]^{-\frac{v+1}{2}} (\sigma_{jt}^2)^{-\frac{1}{2}} (v-2)^{-\frac{1}{2}}, \quad i = a, bt, \quad j = a, b. \quad (2.18)$$

3. Method for Pricing European Options Using Monte Carlo Simulation

The following is a brief explanation of the method for obtaining the option prices by means of the Monte Carlo simulation.

When investors are risk neutral, the price of a European option becomes the present discounted value that is calculated by discounting the expectation of the option price at the maturity with the interest rate of risk-free assets r . Namely, when it is assumed that the T is maturity and that C_t is the price of the call option of the exercise price K at time t and that P_t is the put option price, the following expressions are obtained:

$$C_t = e^{-r(T-t)} E [Max (S_T - K, 0)], \quad (3.1)$$

$$P_t = e^{-r(T-t)} E [Max (K - S_T, 0)]. \quad (3.2)$$

Here, S_T represents the underlying asset price at the maturity of the option.

In the case of the normal mixture EGARCH model, since there is no closed form analytical solution for Eqs. (3.1) and (3.2), these expectations are estimated using Monte Carlo simulation. Let $\{S_T^{(i)}\}_{i=1}^n$ be the simulated values of S_T and n be the number of sample paths. When n is sufficiently large, by the law of large numbers, these expectations can be approximated by the following equations:

$$E [Max (S_T - K, 0)] \approx \frac{1}{n} \sum_{i=1}^n Max (S_T^{(i)} - K, 0), \quad (3.3)$$

$$E [Max (K - S_T, 0)] \approx \frac{1}{n} \sum_{i=1}^n Max (K - S_T^{(i)}, 0). \quad (3.4)$$

The procedures for calculating an option price with the Monte Carlo simulation in this study's model are as follows:

Step 1. Estimate the unknown parameters of the normal mixture EGARCH model with the maximum likelihood method, using the samples $\{R_1, R_2, \dots, R_t\}$.

Step 2. Sample $\{z_{t+1}^{(i)}, z_{t+2}^{(i)}, \dots, z_T^{(i)}\}_{i=1}^n$ from independent standard normal distributions.

Step 3. Sample $\{u_{1,t+1}^{(i)}, u_{1,t+2}^{(i)}, \dots, u_{1,T}^{(i)}\}_{i=1}^n$ and $\{u_{2,t+1}^{(i)}, u_{2,t+2}^{(i)}, \dots, u_{2,T}^{(i)}\}_{i=1}^n$ from independent standard rectangular distributions.

Step 4. Calculate $\{R_{t+1}^{(i)}, R_{t+2}^{(i)}, \dots, R_T^{(i)}\}_{i=1}^n$ by substituting the values at Steps 2. and 3. into the Normal Mixture EGARCH model.

Step 5. Obtain the underlying asset price $(S_T^{(1)}, S_T^{(2)}, \dots, S_T^{(n)})$ at the maturity time T of the option with the following equation:

$$S_T^{(i)} = S_t \prod_{s=1}^{T-t} (1 + R_{t+s}^{(i)}), \quad i = 1, 2, \dots, n. \quad (3.5)$$

Step 6. Calculate the call option's price C_t and the put option's price P_t with the following equation:

$$C_t \approx e^{-r(T-t)} \frac{1}{n} \sum_{i=1}^n \text{Max}(S_T^{(i)} - K, 0), \quad (3.6)$$

$$P_t \approx e^{-r(T-t)} \frac{1}{n} \sum_{i=1}^n \text{Max}(K - S_T^{(i)}, 0). \quad (3.7)$$

It is considered that the sufficient number of times of the Monte Carlo simulation is about 10,000. In order to reduce the variances of C_t and P_t , we propose the method of concurrently using the control variates and the antithetic variates, which are representative variance reduction techniques¹⁾ (see Satoyoshi and Mitsui (2011), Appendix B). In the case of the t mixture EGARCH model, we can use Monte Carlo simulation in the same procedures.

4. Conclusion

In this note, we proposed the normal mixture EGARCH model to simultaneously handle volatility fluctuations over time, the leverage effect and both the thickness of tails and asymmetry for underlying asset rate of return on options. Application of the model to the European option evaluation was also covered. In fact, for Nikkei 225 options if the Nikkei 225 data for the underlying assets is available then the normal mixture EGARCH model and t mixture EGARCH model proposed in this note can estimate option prices.

When it comes to extensions of the model, the use of other models that cover the leverage effect

1) In addition to these techniques, a variety of techniques has been proposed, including the stratified sampling, the Latin hypercube sampling, and the importance sampling.

could be considered, such as the asymmetric GARCH model by Engle and Ng (1993) or the GJR model by Glosten et al. (1993). Moreover, instead of assuming investor risk neutrality, it may be effective to create equations considering the risk premiums in underlying asset rate of return processes covered in studies such as Duan (1995) and Siu *et al.* (2004).

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