

Reversibility of Markov chains with applications to storage models

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abstract

This paper documents some results on reversibility of Markov chains and their applications to queueing and storage models according to the Ōsawa's papers. His research on reversibility started from the 1980's and has been continued to the present. At first, reversibility conditions for Markov chains with general state space were proposed and they were applied to variable Markov models, for example, queueing processes, queueing networks, Markov storage models and so on. Many results have been obtained about these problems, some of them are reviewed in this paper.

Keywords : reversibility, time-reversibility, Markov chain, Markov increment process, quasi-reversibility, queue, queueing network, storage model.

1. Introduction

This paper reviews some results on reversibility of Markov chains and its applications to storage processes according to the Ōsawa's papers. He has studied this theme since 1985.

First, Ōsawa ([2], [5], [6]) investigated reversibility of Markov chains with general state space and obtained the necessary and sufficient conditions that the Markov chain is reversible. Using these results, reversibility of queueing processes was studied by Ōsawa ([2], [3], [6], [12], [13]). Moreover, Ōsawa made research on reversibility of autoregressive processes ([4], [15]), Markov increment processes ([16], [18]) and Markovian storage models ([5], [6], [14]).

In 1970's, Kelly took the leader in developing applications to the wide range of models in operations research, particularly queues and queueing networks, see [1]. He also proposed quasi-reversibility as the property that brings the product-form solution for queueing networks in the same way as reversibility. Ōsawa studied quasi-reversibility for storage processes in discrete-time ([7], [8], [9], [10], [11], [14], [17], [19]). The results obtained in the above papers are collected in this paper.

2. Reversibility

We consider a Markov chain $\mathbf{X} = \{X_n, n \in \mathbb{Z}\}$ with the state space S where Z is the set of all integers. In this paper, "Markov chain" means "Markov process in discrete-time". Let (S, \mathcal{F}, P) be a probability space and assume that X has transition probabilities $P(x, A)$, i.e.,

$$P(x, A) = P[X_{n+1} \in A \mid X_n = x], \text{ for } x \in S \text{ and } A \in \mathcal{F}.$$

In this section, we describe the conditions under that a Markov chain X is reversible in accordance with Ōsawa ([2], [6]).

2.1 Definition

If there exists a measure ν satisfying that

$$\text{(Rev1)} \quad \int_A \nu(dx)P(x, B) = \int_B \nu(dx)P(x, A), \text{ for any } A, B \in \mathcal{F},$$

then a Markov chain X is said to be reversible and ν is called a reversible measure.

Remark 2.1 Consider the Markov chain X on the real line R with transition densities

$$p(x, y), x, y \in R, \text{ i.e., } P(x, A) = \int_A p(x, y)dy.$$

Then (Rev 1) is rewritten as

$$p(x)p(x, y) = p(y)p(y, x), \text{ for any } x, y \in R,$$

where $p(\cdot)$ is a measurable function on R .

Remark 2.2 Consider the Markov chain X on Z with transition probabilities

$$p(i, j), i, j \in Z, \text{ i.e., } P(x, A) = \sum_{j \in A} p(i, j).$$

Then (Rev 1) is rewritten as

$$p_i p(i, j) = p_j p(j, i), \text{ for any } i, j \in Z,$$

where $\{p_i\}$ is a measurable function on Z .

In this section, we describe the conditions under that a Markov chain X is reversible in accordance with Ōsawa ([2], [6]). Let $B(S)$ be the set of real-valued \mathcal{F} -measurable bounded functions defined on S , then the following notations are used:

$$\nu f = \int_S \nu(dx)f(x),$$

$$Pf(x) = \int_S P(x, dy)f(y),$$

for any measurable functions $f(\cdot)$ on S .

Lemma 2.1 (Ōsawa [2], [6]) We have followings.

(1) (Rev1) is equivalent to

$$\nu(f_1 Pf_2) = \nu(f_2 Pf_1), \text{ for any } f_1, f_2 \in B(S).$$

(2) For a reversible Markov chain, the reversible measure ν is invariant, i.e.,

$$\int_S \nu(dx)P(x,A) = \nu(A), \text{ for any } A \in F.$$

2.2 Time-reversibility

For a stochastic process $\{Y_n, n \in \mathbb{Z}\}$, if the time-reversed process $\{Y_{-n}, n \in \mathbb{Z}\}$ has the same probabilistic law, that is,

$$(Y_{n_1}, Y_{n_2}, \dots, Y_{n_k}) = (Y_{m-n_1}, Y_{m-n_2}, \dots, Y_{m-n_k}) \text{ with probability 1,}$$

for any integers $m, n_1 < n_2 < \dots < n_k$, then $\{Y_n\}$ is said to be time-reversible. It is clear that the time-reversible process is stationary (see Kelly [1]). We have a following theorem on time-reversibility of Markov chains X .

Theorem 2.2 (Osawa [2], [6]) *A stationary Markov chain X is time-reversible if and only if X is reversible with the bounded measure ν . And then the probability measure*

$$\pi(A) = \frac{\nu(A)}{\nu(S)}$$

is the stationary distribution of X .

2.3 Markov chains with atoms

Let δ be a state such that

$$P(x, \{\delta\}) > 0, \text{ for some } x \in S,$$

then δ is said to be an atom. In this section, we describe reversibility of Markov chains with atoms.

Theorem 2.3 (Osawa [2], [6]) *Suppose that a Markov chain X has an atom such as*

$$P(x, \{\delta\}) > 0, \text{ for any } x \in S_\delta \equiv S - \{\delta\}.$$

Then X is reversible if and only if the transition probability P satisfies

$$(Rev2) \quad P(f_1 P(f_2 P_{1_\delta}))(\delta) = P(f_2 P(f_1 P_{1_\delta}))(\delta),$$

for any $f_1, f_2 \in B(S)$ where 1_δ is the indicator of $\{\delta\}$.

In the above theorem, a reversible measure is given by

$$\nu_\delta(dx) = \frac{P(\delta, dx)}{P(x, \{\delta\})}, \quad x \in S.$$

Thus if ν_δ is bounded, X is time-reversible.

Remark 2.3 If the Markov chain X in Remark 2.1 has an atom δ in Theorem 2.3, then (Rev 2) is rewritten as

$$p(\delta, x)p(x, y)P(y, \{\delta\}) = p(\delta, y)p(y, x)P(x, \{\delta\}), \quad x, y \in S_\delta.$$

We now consider the Markov chain with several atoms. Let δ_i be the atoms and denote the set of all atoms by Δ , that is,

$$\Delta = \{\delta_1, \delta_2, \dots, \delta_m\}.$$

Theorem 2.4 (Ōsawa [2], [6]) *Suppose that a Markov chain X has the set of atoms as above and the transition probability P satisfying*

$$P(x, \{\delta_i\}) > 0, x \in S - \{\delta_i\}, i = 1, 2, \dots, m.$$

Then X is reversible if and only if there exists constants $c_i > 0$ such that

$$c_i P(f_1 P(f_2 P 1_{\delta_j})) (\delta_i) = c_j P(f_2 P(f_1 P 1_{\delta_i})) (\delta_j),$$

for any $i, j = 1, 2, \dots, m$, and $f_1, f_2 \in B(S)$.

Remark 2.4 Consider the Markov chain X in Remark 2.2 having a state such that $p(i, \delta) > 0$ for any $i \in Z$. Then X is reversible under the condition;

$$p(\delta, i)p(i, j)p(j, \delta) = p(\delta, j)p(j, i)p(i, \delta), i, j \in Z.$$

3 Markov increment processes

Let $\{U_n, n \in Z\}$ be a sequence of independent identically distributed (*i.i.d.*) random variables. Consider a stochastic process $\{X_n, n \in Z\}$ defined by

$$X_{n+1} = g(X_n + U_n), n \in Z,$$

where $g(x)$ is a measurable function on the measurable space (S, F) . Clearly, the process $\{X_n\}$ is a Markov chain and said to be a Markov increment process (*MIP*).

3.1 *MIP* on the real line

Suppose that the state space is the real line, $S = R$, or it's subset and $\{U_n\}$ has a distribution function $U(x)$ with a density function $u(x)$. We make following assumptions;

- $u(x)$ is continuous at $x = 0$.
- $u(x)$ is differentiable at $x = 0$ from the right and the left.

In this paper, we describe time-reversibility of following processes.

(C1) Markov Additive Process (*MAP*)

*) Consider the function $g(x) = x$, then the process $\{X_n\}$, defined by

$$X_{n+1} = X_n + U_n, \text{ is called the Markov additive process (MAP).}$$

*) It has transition densities $p(x, y) = u(y - x)$, $i, j \in R$.

(C2) Lindley Process (*LP*)

*) The Lindley process is a kind of *MIP* defined by the function

$$g(x) = \max(0, x).$$

*) The state space is $R_0 = [0, \infty)$ and the state 0 is only one atom.

*) Transition probabilities are given by

$$\begin{aligned} p.d.f. p(x, y) &= u(y - x), \quad x \geq 0, y > 0, \\ P(x, \{0\}) &= U(-x), \quad x \geq 0. \end{aligned}$$

(C3) Bounded Lindley Process (BLP)

*) Let K be a positive constant and consider the function

$$g(x) = \min(\max(0, x), K).$$

*) The state space is $R_B = [0, K]$ and there are two atoms, 0 and K .

*) Suppose that $u(x) = 0$ for $|x| > K$.

*) Transition probabilities are given by

$$\begin{aligned} p.d.f. p(x, y) &= u(y - x), \quad 0 \leq x \leq K, 0 < y < K, \\ P(x, \{0\}) &= U(-x), \quad 0 \leq x < K, \\ P(x, \{K\}) &= 1 - U(K - x), \quad 0 \leq x \leq K. \end{aligned}$$

We have followings for reversibility of the above processes.

Theorem 3.1

(1) (Osawa [2], [6]) *LP process is reversible if and only if $U(x)$ is the two-sided exponential distribution given by*

$$u(x) = \begin{cases} c \exp(ax), & x < 0, \\ c \exp(-bx), & x \geq 0. \end{cases}$$

where $a, b > 0$ and $c = \frac{ab}{a+b}$. Moreover, if $a < b$ then LP is time-reversible.

(2) (Osawa and Doi [16]) *BLP process is time-reversible if and only if $U(x)$ is the two-sided exponential distribution with barriers given by*

$$\begin{aligned} u(x) &= \begin{cases} c \exp(ax), & -K < x < 0, \\ c \exp(-bx), & 0 \leq x < K, \end{cases} \\ U(-K) &= \frac{c}{a} \exp(-aK), \\ U(K) &= \frac{c}{b} \exp(-bK), \end{aligned}$$

where $a, b > 0$ and $c = \frac{ab}{a+b}$.

Remark 3.1 If $u(x)$ is the symmetric probability density with respect to $x = 0$, then the MIP process is reversible but not time-reversible. Thus there are not time-reversible MIP processes.

3.2 MIP with discrete-states

Suppose that the state space is the set of all integers or it's subset and U_n has a probability function $\{u_k, k \in Z\}$. In this section, we describe time-reversibility of following processes.

(D1) Markov Additive Process (*DMAP*)

*) Consider the function $g(x) = x$, then the process $\{X_n\}$, defined by

$X_{n+1} = X_n + U_n$, is written as *DMAP* process.

*) It has transition functions $p(i, j) = u_{j-i}$, $i, j \in \mathcal{N}$.

(D2) Lindley Process (*DLP*)

*) The Lindley process is a kind of *MIP* defined by the function

$$g(x) = \max(0, x).$$

*) The state space is the set of all non-negative integers $\mathcal{N}_0 = \{0, 1, \dots\}$.

*) Transition probabilities are given by

$$p(i, j) = \begin{cases} u_{j-i}, & i \in \mathcal{N}_0, j > 0, \\ U(-i) = \sum_{k=-\infty}^{-i} u_k, & i \in \mathcal{N}_0, j = 0. \end{cases}$$

(D3) Bounded Lindley Process (*DBL*)

*) Let $J \geq 3$ be a positive integer and consider the function

$$g(x) = \min(\max(0, x), J).$$

*) The state space is $\mathcal{N}_B = \{0, 1, 2, \dots, J\}$.

*) Suppose that $u = i = 0$ for $|i| > J$.

*) Transition probabilities are given by

$$p(i, j) = \begin{cases} u_{j-i}, & i \in \mathcal{N}_B, j \in \mathcal{N}_B - \{0\}, \\ U(-i) = \sum_{k=-J}^{-i} u_k, & i \in \mathcal{N}_B, j = 0. \end{cases}$$

Theorem 3.2

(1) (\bar{O} sawa [2], [6], \bar{O} sawa and Shima [12]) *DLP process is reversible if and only if $\{u_k, k \in Z\}$ is the two-sided geometric distribution given by*

$$u_k = \begin{cases} ap^{k-1}, & k = 1, 2, \dots, \\ c, & k = 0, \\ bq^{-k-1}, & k = -1, -2, \dots, \end{cases}$$

where $0 < p, q < 1$, $a, b > 0$ and $c \geq 0$ satisfy

$$\frac{a}{1-p} + c + \frac{b}{1-q} = 1 \quad \text{and} \quad aq = bp.$$

Moreover, if $p < q$ then DLP is time-reversible.

(2) (Osawa and Zhang [18]) DBLP process is time-reversible if and only if $\{u_k, k \in \mathcal{N}_B\}$ is the distribution given by

$$u_k = \begin{cases} ap^{k-1}, & k = 1, 2, \dots, J-1, \\ c, & k = 0, \\ bq^{-k-1}, & k = -1, -2, \dots, -J+1, \end{cases}$$

$$u_{-J} = U(-J) = \frac{b}{1-q} q^{J-1},$$

$$u_J = U(J) = \frac{a}{1-p} p^{J-1},$$

where $0 < p, q < 1$, $a, b > 0$ and $c \geq 0$ satisfy

$$\frac{a}{1-p} + c + \frac{b}{1-q} = 1.$$

Remark 3.2 (Osawa and Zhang [18]) In DBLP process, if $J = 1$ the process is time-reversible.

Moreover, in the case of $J = 2$ the process is time-reversible if and only if the following relations holds;

$$u_1(u_1 + u_2)u_{-2} = u_2(u_{-1} + u_{-2})u_{-1}.$$

Remark 3.3 If $\{u_i, i \in \mathcal{Z}\}$ is the symmetric distribution with respect to $i = 0$, then DMAP process is reversible but not time-reversible. Thus there are not time-reversible DMAP processes.

4. Reversibility of queueing processes

In this section, we describe reversibility of following queueing processes.

[1] Waiting-time process in the ordinary queue

Consider the single server queue GI/GI/1. Let A_n be the service time of n th customer and B_n be the interarrival time between n th and $(n+1)$ th customers. Assume that $\{A_n\}$ is a sequence of *i.i.d.* random variables with the common distribution function $A(x)$. And $\{B_n\}$ is also a sequence of *i.i.d.* random variables with the common distribution function $B(x)$. Let $U_n = B_n - A_n$, then the waiting-time process is formulated by the LP process. Applying Theorem 3.1(1) yields that the waiting-time process in GI/GI/1 is reversible if and only if both $A(x)$ and $B(x)$ are exponential

distributions, that is, M/M/1 (\bar{O} sawa [2], [6]).

[2] Waiting-time process in the discrete-time queue

Consider the discrete-time queue GI/GI/1. Assume that the service time A_n of n th customer has a distribution $\{a_k, k \in \mathcal{N}_0\}$, and the interarrival time between n th and $(n + 1)$ th customers B_n has a distribution $\{b_k, k \in \mathcal{N}_0\}$. By letting $U_n = B_n - A_n$, the waiting-time process $\{X_n\}$ is described by *DLP*. And we have the following.

For the waiting-time process in the discrete-time GI/GI/1 queues, the reversible one is only for the M/M/1, that is, $\{a_k\}$ and $\{b_k\}$ are geometric distributions (\bar{O} sawa [12], [13]).

[3] Quasi-reversibility

In the above systems, we can see the property that the departure process is the same probability law as the arrival process. This is called the quasi-reversibility. This property has the essential role that the system has the product-form solution in queueing networks with time-reversible nodes.

5. Quasi-reversibility

In the above sections, we found that there are few systems having time-reversibility. However, it is known that there are many systems having quasi-reversibility (\bar{O} sawa [6], [7], [8], [10], [17], [19]).

Consider the discrete-time storage model with discrete states as follows. Let A_n be the number of arriving items just after time n , and D_{n+1} be the number of departures at the end of n th slot. Then the system states X_n , the number of items at n th time epoch, is defined by

$$X_{n+1} = X_n + A_n - D_{n+1}$$

Suppose that D_{n+1} depends only on the system state just after n th time epoch, i.e., $X_n + A_n$,

$$\begin{aligned} P[D_{n+1} = j | X_n, A_n, D_m, m \leq n] \\ = P[D_{n+1} = j | X_n + A_n = i] \\ = D(i, j), \quad 0 \leq j \leq i, i \geq 0. \end{aligned}$$

Consider the departure rule

$$D(i, j) = c(i)r(i - j)\gamma(j), \quad 0 \leq j \leq i, i \geq 0,$$

where $r(j), \gamma(j)$ are non-negative real-valued functions and $c(i)$ is determined by

$$c(i) = \left\{ \sum_{j=0}^i r(i - j)\gamma(j) \right\}^{-1}, \quad i \geq 0.$$

Further, we assume that $r(j) > 0$ for $j \geq 0$ and $\sum r(j) < \infty$. This departure rule is called

D-rule. Then we have the following.

Theorem 5.1 (Ōsawa [9], [19]) *The discrete-time storage model with D-rule has the following properties;*

- *the past of the departure process $\{D_k, k \leq n\}$ is independent of the present state X_n ,*
- *the future of the arrival process $\{A_k, k \geq n\}$ is independent of the present state X_n ,*
- *the departure D_n and the arrival A_n have the same distribution.*

The system having the properties in Theorem 5.1 is said to be quasi-reversible, see Kelly [1].

Moreover, we have the following.

Theorem 5.2 (Ōsawa [9], [19]) *The discrete-time storage model has quasi-reversibility if and only if the departure rule is D-rule.*

6. Comments

This paper reviewed some results on reversibility of Markov chains with applications to Markov models, queues, queueing networks, storage models, and so on. Further, some new results are introduced and these works are in preparation. The research on this theme has been continuing.

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