Admissible vertices of contraction-critically 5-connected graphs

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abstract

Let G be a 5-connected graph. An edge of a G is said to be 5-contractible if the contraction of the edge results in a 5-connected graph. If G has no 5-contractible edge, then it is said to be contraction-critical. An induced subgraph A of G is said to be a fragment if $|N(A)|=5$ and $V(G)-(A\cup N(A))\neq\emptyset$, where $N(A)$ is the neighborhood of A. For a fragment A and $x\in N(A)$, a vertex $z\in N(x)\cap N(A)$ is said to be an admissible vertex for $(x;A)$, if the degree of z is 5 and $|N(z)\cap A|\geq 2$. We show some new properties on admissible vertices of contraction-critically 5-connected graphs. Using admissible vertices, we give a result on the structure around a fragment whose cardinality is 2.

Key Words: 5-connected graph, contraction-critically 5-connected, degree 5 vertex AMS classification: 05C40

Dedicated to Professor Hideo Osawa on the occasion of his retirement.

1 Introduction

In this paper, we deal with finite undirected graphs with neither self-loop nor multiple edge. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G, respectively. We call $|V(G)|$ and $|E(G)|$ the order of G and the size of G, respectively. Let $V_k(G)$ denote the set of vertices of degree k. For an edge $e\in E(G)$, we denote the set of end vertices of e by $V(e)$. For a vertex $x\in V(G)$, we denote by $N_G(x)$ the neighborhood of x in G. Moreover, for a subset $S\subseteq V(G)$, let $N_G(S)=\cup_{x\in S}N(x)-S$. We denote the degree of $x\in V(G)$ by $\deg_G(x)$. For a vertex $x\in V(G)$, we denote by $E_G(x)$ the set of edges incident with x. Then $\deg_G(x)=|N(x)|=|E_G(x)|$. When there is no ambiguity, we write V_k, $N(x)$, $N(S)$, $\deg(x)$ and $E(x)$ for $V_k(G)$, $N_G(x)$, $N_G(S)$, $\deg_G(x)$ and $E_G(x)$, respectively. For $S\subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S in G. For $S\subseteq V(G)$, we let $G-S$ denote the graph obtained from G by deleting the vertices in S together with the edges incident with them; thus $G-S = G[V(G)-S]$. A subset $S \subseteq V(G)$ is said to be a ***cutset*** of G, if $G-S$ is not connected. A cutset S is said to be a ***k-cutset*** if $|S|=k$. For a noncomplete connected graph G, the order of a minimum cutset of G is said to be the connectivity of G denoted by $\kappa(G)$. Let G be a connected graph with $\kappa(G)=k$. We denote by K_n the complete graph on n vertices. For graphs G and H, we write $G+H$ the join of G and H.

Let k be an integer such that $k \geq 2$ and let G be a k-connected graph with $|V(G)| \geq k+2$. An edge e of G is said to be ***k-contractible*** if the contraction of the edge results in a k-connected graph. Note that, in the contraction, we replace each resulting pair of double edges by a simple edge. If an edge is not k-contractible, then it is called ***k-noncontractible***. Note that an edge e of G is k-noncontractible if and only if there is a k-cutset S of G such that $V(e) \subseteq S$. If G does not have a k-contractible edge, then G is said to be ***contraction-critically k-connected***.

An induced subgraph A of G is called a ***fragment*** if $|N(A)|=k$ and $V(G)-(A \cup N(A)) \neq \emptyset$. If $|A|=i$, then a fragment A is called ***i-fragment***. A noncontractible edge e is said to be ***trivial***, if there is a fragment A such that $|A|=1$ and $V(e) \subseteq N(A)$. A noncontractible edge e is said to be ***far from trivial***, if $|A| \geq \frac{1}{2}(|V(G)|-2k)$ for any fragment A such that $V(e) \subseteq N(A)$.

Let G be a 5-connected graph. Let $x \in V(G)$ and let A be a fragment of G such that $x \in N(A)$. For $y \in N(x) \cap A$, a vertex z is said to be an ***admissible vertex for*** $(x,y;A)$, if $z \in N(x) \cap N(y) \cap S \cap V_5$ and $|N(z) \cap A| \geq 2$. A vertex z is said to be an ***admissible vertex for*** $(x;A)$, if z is an admissible vertex for $(x,y;A)$ for some $y \in N(x) \cap A$.

It is known that every 3-connected graph of order 5 or more contains a 3-contractible edge [13]. There are infinitely many contraction-critically k-connected graphs for each $k \geq 4$ [12]. It is known that a 4-connected graph G is contarction-critical if and only if G is 4-regular, and for each edge e of it, there is a triangle which contains e [8, 10].

Egawa determined the following sharp minimum degree condition for a k-connected graph to have a k-contractible edge.

Theorem A (Egawa [7]) *Let k be an integer, let G be a k-connected graph with $\delta(G) \geq \left[\frac{5n}{2}\right]$. Then G has a k-contractible edge, unless $2 \leq k \leq 3$ and G is isomorphic to K_{k+1}.*

There are infinitely many contraction-critically 5-connected graphs which are not 5-regular. However, by virtue of Theorem A, we know that the minimum degree of a contraction-critically 5-connected graph is 5.

The following result due to Su says that there are degree 5 vertices everywhere in a contraction-critically 5-connected graph.

Theorem B (Su [11]) *Every vertex of a contraction-critically* 5*-connected graph has two neighbors of degree five.*

Since a contraction-critically 4-connected graph is 4-regular, it has very restricted substructure. On the other hand, for any given graph, there is a contraction-critically 5-connected graph which has it as an induced subgraph.

Theorem C (Ando and Kawarabayashi [6]) *Let k be an integer such that* $k \geq 5$ *and let H be a graph. Then, we can construct a contraction-critically k-connected graph which contains H as an induced subgraph.*

Theorem C indicates the big difference between 'contraction-critically 4-connected graphs' and 'contraction-critically 5-connected graphs'. As Kriesell wrote in [9], it is probably a tremendously hard problem to characterize contraction-critically *k*-connected graphs for $k \geq 5$. Although we still do not have enough knowledge of the global structure of contraction-critically 5-connected graphs, we have a local structure theorem on contraction-critically 5-connected graphs [1] and we also have some progress on the study of contraction-critically 5-connected graphs [3, 4, 5, 6]. In the last decade, in the study of contraction-critically 5-connected graphs, 'admissible vertices' play crucial roles. In this paper we focus on admissible vertices of contraction-critically 5-connected graphs and we show some new conditions for a contraction-critically 5-connected graph to have an admissible vertex. Furthermore, using admissible vertices, we prove the following Theorem 1 which shows the remarkable structure around a connected 2-fragment of a contraction-critically 5-connected graph.

Theorem 1 *Let G be a contraction-critically* 5*-connected graph. Let A be a connected fragment of G with* $|A| = 2$, *say* $A = \{x_1, x_2\}$ *and let* $S = N(A)$.

(1) *If* $A \cap V_6 \neq \emptyset$, *then the number of vertices* $y \in S$ *such that there is an admissible vertex for* $(y; A)$ *is greater than or equal to* 4.

(2) *If* $A \cap V_6 \neq \emptyset$, *then the number of admissible vertices for some* $(y; A)$ *is greater than or equal to* 3.

(3) If $A \cap V_6 = \emptyset$, *then there is a vertex* $y \in S - N(x_1) \cap N(x_2)$ *such that there is an admissible vertex for* $(y; A)$.

This paper consists of 4 sections. After presenting preliminary results in section 2, we give some sufficient conditions for the existence of admissible vertices for given pair (x, A), where A is a fragment of a contraction-critically 5-connected graph and $x \in N(A)$, in section 3. In section 4, we give a proof of Theorem 1.

To conclude the section, we present three contraction-critically 5-connected graphs. The first one is 5-regular, and for each edge e of it, there is a triangle which contains e. Hence, this graph is similar in structure to contraction-critically 4-connected graphs. The second one has large maximum degree. The last one has an edge which is far from trivial. We observe that every edge in a contraction-critically 4-connected graph is trivial and, every edge of the first example and the second example is trivial. However the number of non-trivial noncontractible edges of the last example is proportional to the size of it.

Example 1

Identifying the top and the bottom, and the left side and the right side of the graph in Fig 1, we obtain a 5-regular contraction-critically 5-connected graph for each edge e of which, there is a triangle containing e.

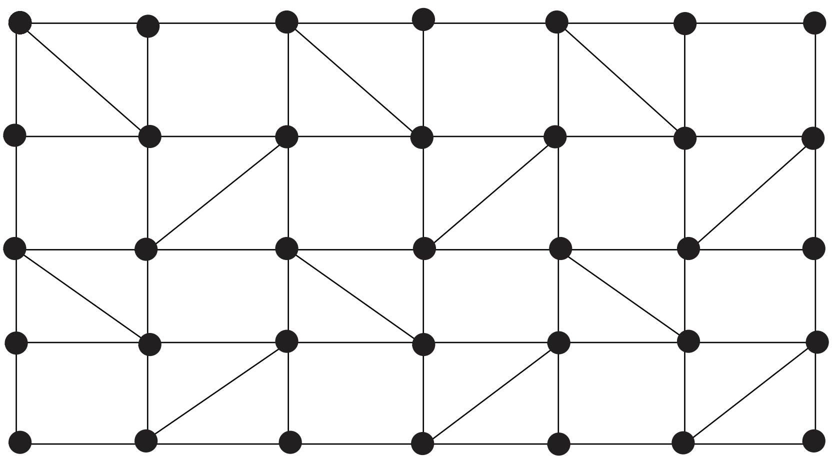

Fig.1: A contraction-critically 5-connected graph similar in structure to contraction contraction-critically 4-connected graphs

Example 2

Let H be a contraction-critically 4-connected graph and let $G = H + K_1$. Then, we observe that G is 5-connected and every edge of G is trivially 5-noncontractible. Hence G is a contraction-critically 5-connected graph with $\Delta(G) = |V(G)| - 1$.

Example 3

Let K_4^- stand for the graph obtained from K_4 by removing one edge; that is $K_4^- \cong K_2 + 2K_1$. Let m be an integer such that $m \geq 3$ and we construct $G^{(m)}$ as follows; At first we prepare a configuration H_m which consists of m copies of K_4^- (see in Fig.2). Next take other three distinct vertices and join them to bottom part vertices of H_m. At last take two distinct K_4^-'s and join one K_4^- to the left side 2 vertices of H_m and the three distinct vertices, and join the other K_4^- to the right side 2 vertices of H_m and the three distinct vertices, appropriately (see Fig.3).

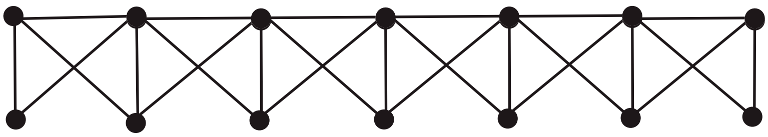

Fig.2: H_6

We call the resulting graph $G^{(m)}$. Let e be an edge of the top part of H_m. Then we observe that there is a 5-cutset of $G^{(m)}$ consisting of $V(e)$ and the distinct three vertices. Moreover, we observe that this is the only 5-cutset in $G^{(m)}$ which contains $V(e)$. By these observations, we know that $G^{(m)}$ is a contraction-critically 5-connected graph and it has a far from trivial edge and many non-trivial 5-noncontractible edges.

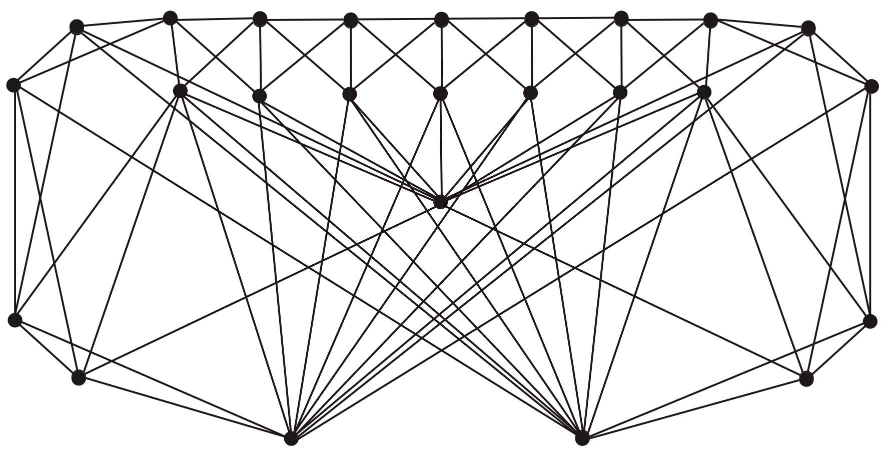

Fig.3: $G^{(6)}$: contraction-critically 5-connected graph with many non-trivial edges

2 Preliminaries

In this section we give some more definitions and preliminary results.

For a graph G, we denote $|G|$ for $|V(G)|$. For a subgraphs A and B of a graph G, when there is no ambiguity, we write simply A for $V(A)$ and B for $V(B)$. So $N(A)$ and $A \cap B$ mean $N(V(A))$ and $V(A) \cap V(B)$, respectively. Also for a subgraph A of G and a subset S of $V(G)$ we write $A \cap S$ and $A \cup S$ for $V(A) \cap S$ and $V(A) \cup S$, respectively. When there is no ambiguity, we write $E(S)$ for $E(G[S])$. For subset S and T of $V(G)$, we denote the set of edges between S and T by $E_G(S,T)$. We write $E_G(x,T)$ for $E_G(\{x\,T\})$. When there is no ambiguity, we write $E(S,T)$ and $E(x,T)$ for $E_G(S,T)$ and $E_G(x,T)$, respectively. Let $V_{\geq k}(G)$ (or sometimes simply $V_{\geq k}$) denote the set of vertices of degree at least k.

Let G be a connected graph with $\kappa(G) = k$. Recall that an induced subgraph A of G is called a fragment if $|N(A)| = k$ and $V(G) - (A \cup N(A)) \neq \emptyset$. In other words, a fragment A is a nonempty union of components of $G - S$ where S is a k-cutset of G such that $V(G) - (A \cup S) \neq \emptyset$. By the definition, if A is a fragment of G, then $G - (A \cup N(A))$ is also a fragment of G. Let $\bar{A}$ stand for $G - (A \cup N(A))$. For an edge e of G, a fragment A of G is said to be a *fragment with respect* to e if $V(e) \subseteq N(A)$. For a set of edges $F \subseteq E(G)$, we say that A is a *fragment with respect to F* if A is a fragment with respect to some $e \in F$. A fragment A with respect to F is said to be *minimum* (resp. *minimal*) if there is no fragment B other than A with respect to F such that $|B| < |A|$ (resp. $B \subsetneq A$). If $|A| = 1$, then a fragment A is said to be *trivial*.

Let $V_k^{(i)}(G)$ (or sometimes simply $V_k^{(i)}$) stand for the set of vertices of $V_k(G)$ each of which has i neighbors in $V_k(G)$, namely $V_k^{(i)} = \{x \in V_k(G) \,||\, N(x) \cap V_k(G) |= i\}$.

We start with the following Lemma 1 [3] which is a simple but useful observation. We give a proof of Lemma 1 for the reader's convenience.

Lemma 1 *Let A be a fragment of a k-connected graph G and let* $S \subseteq N(A)$. *If* $|N(S) \cap A| < |S|$, *then* $A = N(S) \cap A$.

Proof. Assume that $A \neq N(S) \cap A$. Let $A' = A - (N(S) \cap A)$. Since $A' \neq \emptyset$ and $T \cap (\bar{A} \cup S) = \emptyset$, $(N(A) - S) \cup (N(S) \cap A)$ separates A' and $\bar{A} \cup S$. Since $|N(S) \cap A| < |S|$, we see that $|(N(A) - S) \cup (N(S) \cap A) = |N(A)| - |S| + |N(S) \cap A| < |N(A)| = k$, which contradicts the k-connectedness of G. ■

The reader can find the proof of Lemma 2 in [3].

Lemma 2 *Let G be a 5-connected graph, and let A and B be fragments of G. Let* $S = N(A)$ *and let* $T = N(B)$.

	$\bar{A}$	S	A
B	$\bar{A} \cap B$	$S \cap B$	$A \cap B$
T	$\bar{A} \cap T$	$S \cap T$	$A \cap T$
$\bar{B}$	$\bar{A} \cap \bar{B}$	$S \cap \bar{B}$	$A \cap \bar{B}$

Then the following hold.

(1) *If* $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geq 6$, *then* $|(\bar{A} \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| \leq 4$ *and* $\bar{A} \cap \bar{B} = \emptyset$. *In particular, if neither* $A \cap B$ *nor* $\bar{A} \cap \bar{B}$ *is empty, then both* $A \cap B$ *and* $\bar{A} \cap \bar{B}$ *are fragments of G.*

(2) $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = 5 + |S \cap B| - |\bar{A} \cap T|$. *In particular, if* $A \cap B \neq \emptyset$, *then* $|S \cap B| \geq |\bar{A} \cap T|$.

(3) *If* $|\bar{A}| \geq 2$, *then either* $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$ *or* $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \leq 5$.

3 Admissible vertices

In the following two sections we consider 5-connected graphs.

We introduce ‘admissible vertex’ in [3] and we introduce ‘strongly admissible vertex’ and ‘hyper admissible vertex’ in [2]. In this paper, we introduce ‘insufficient’ and give a new sufficient condition a contraction-critically 5-connected graph to have an admissible vertex.

Let G be a 5-connected graph. Let $x \in V(G)$ and let A be a fragment of G such that $x \in N(A)$. Let $S = N(A)$.

Let $y \in N(x) \cap A$. Recall that a vertex z is said to be an admissible vertex for $(x, y; A)$, if the following two conditions hold.

(1) $z \in N(x) \cap N(y) \cap S \cap V_5$.

(2) $|N(z) \cap A| \geq 2$.

Here, we introduce more detailed properties of admissible vertices.

For $y \in N(x) \cap A$, a vertex z is said to be an *strongly admissible vertex for* $(x, y; A)$, if the following conditions hold.

(1) $z \in N(x) \cap N(y) \cap S \cap V_5$,

(2) $|N(z) \cap A| \geq 2$, and

(3) $|N(z) \cap \bar{A}| = 1$.

For $y \in N(x) \cap A$, a vertex z is said to be an ***hyper admissible vertex for*** $(x, y; A)$, if the following conditions hold.

(1) $z \in N(x) \cap N(y) \cap S \cap V_5$,

(2) $|N(z) \cap A| \geq 2$, and

(3) $|N(z) \cap \bar{A}| = |N(z) \cap S| = 1$.

A vertex z is said to be a ***strongly admissible vertex for*** $(x; A)$ or a ***hyper admissible vertex for*** $(x; A)$, if z is a strongly admissible vertex for $(x, y; A)$ or a hyper admissible vertex for $(x, y; A)$ for some $y \in N(x) \cap A$, respectively.

A triangle H of G is said to be an *A-inner* x^**-triangle* if (1) $x \in V(H)$, (2) $V(H) - \{x\} \subseteq A$ and (3) $(V(H) - \{x\}) \cap V_5 \neq \emptyset$.

A vertex x is said to be ***insufficient on A*** if the following two conditions hold.

(1) there is no A-inner x^* -triangle.

(2) $N(u) \cap N(u') \cap A \cap V_5 = \emptyset$ for any $u, u' \in N(x) \cap A \cap V_5$.

The following Lemmas 3 and 4 give some basic properties of admissible vertices in a contraction-critically 5-connected graph. The reader can find proofs of Lemmas 3 and 4, and Corollary 7 in [3], however for the convenience of the reader, we give proofs of Lemmas 3 and 4. We give an alternate proof of Corollary 7 in this section.

Lemma 3 ([3] Corollary 4) *Let x be a vertex of a contraction-critically 5-connected graph G. Let A be a fragment of G such that $x \in N(A)$. Suppose $|\bar{A}| \geq 2$, $|A| \geq 3$ and $|N(x) \cap A| = 1$. Then, there is an admissible vertex for $(x; A)$*

Proof. Let $N(x) \cap A = \{y\}$. Let B be a fragment with respect to xy. Let $S = N(A)$ and let $T = N(B)$. Since $|\bar{A}| \geq 2$, by Lemma 2 (3), we see that either $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leq 5$ or $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \leq 5$. Without loss of generality we may assume $|(S \cap B) \cup (S \cap T) \cup$ $\cup (A \cap T)| \leq 5$. Then, since $N(x) \cap A = \{y\}$, we have $A \cap B = \emptyset$.

Claim 3.1 $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \geq 6$.

Proof. If $A \cap \bar{B} \neq \emptyset$, then $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| \geq 6$ since $N(x) \cap A = \{y\}$. Hence, we assume $A \cap \bar{B} = \emptyset$. Then, since $A \cap B = \emptyset$, we have $A = A \cap T$ and $|A| = |A \cap T| \geq 3$, which implies that $|A\cap T|>|S\cap B|$ since $|(S\cap B)\cup(S\cap T)\cup(A\cap T)|\leq 5$.

Hence we observe that $|(S\cap\bar{B})\cup(S\cap T)\cup(A\cap T)|>|S|=5$ and Claim 3.1 is proved. ∎

Claim 3.1 assures us that $|A\cap T|>|S\cap B|$. If $|S\cap B|\geq 2$, then $|A\cap T|\geq 3$ and $|(S\cap B)\cup(S\cap T)\cup(A\cap T)|\geq 6$, which contradicts the fact that $|(S\cap B)\cup(S\cap T)\cup(A\cap T)|\leq 5$. Hence $|S\cap B|\leq 1$. Claim 3.1 also assures us that $\bar{A}\cap B=\emptyset$ and $B=S\cap B$. Let $B=S\cap B=\{z\}$. Then we observe that z is an admissible vertex for $(x;A)$. ∎

Lemma 4 ([3] Lemma 3) Let x be a vertex of a contraction-critically 5-connected graph G. Let A be a fragment such that $x\in N(A)$, $|\bar{A}|\geq 2$ and $|A|\geq 3$. Then, for each vertex $y\in N(x)\cap A$, there is either an admissible vertex for $(x,y;A)$ or a fragment A' with respect to xy such that $A'\subsetneq A$.

Proof. Assume that there is neither an admissible vertex for $(x,y;A)$ nor a fragment A' with respect to xy such that $A'\subsetneq A$. Let B be a fragment with respect to xy. Let $S=N(A)$ and let $T=N(B)$. Since $|\bar{A}|\geq 2$, by Lemma 2 (3), we see that either $|(S\cap B)\cup(S\cap T)\cup(A\cap T)|\leq 5$ or $|(S\cap\bar{B})\cup(S\cap T)\cup(A\cap T)|\leq 5$. Without loss of generality we may assume $|(S\cap B)\cup(S\cap T)\cup(A\cap T)|\leq 5$. If $A\cap B\neq\emptyset$, then $A\cap B$ is a fragment with respect to xy such that $A\cap B\subsetneq A$ since $y\in A\cap T$, which contradicts the assumption. Hence $A\cap B=\emptyset$. ∎

Claim 4.1 $A\cap\bar{B}\neq\emptyset$.

Proof. Assume $A\cap\bar{B}=\emptyset$. Then $A=A\cap T$ and $|A\cap T|=|A|\geq 3$. Hence $|S\cap B|=|(S\cap B)\cup(S\cap T)\cup(A\cap T)|-|S\cap T|-|A\cap T|\leq 5-1-3=1$. Thus $|S\cap B|=1$, say $S\cap B=\{z\}$. Then, we find that z is an admissible vertex for $(x,y;A)$, which contradicts the assumption.

By Claim 4.1, we know that $A\cap\bar{B}\neq\emptyset$. Hence, if $|(S\cap\bar{B})\cup(S\cap T)\cup(A\cap T)|=5$, then $A\cap\bar{B}$ a fragment with respect to xy such that $A\cap\bar{B}\subsetneq A$, which contradicts the assumption. Thus we have $|(S\cap\bar{B})\cup(S\cap T)\cup(A\cap T)|\geq 6$, which implies $\bar{A}\cap B=\emptyset$ and $|S\cap B|<|A\cap T|$. Therefore, $B=S\cap B$ and $|S\cap B|=|(S\cap B)\cup(S\cap T)\cup(A\cap T)|-|S\cap T|-|A\cap T|\leq 4-|A\cap T|<4-|S\cap B|$. Hence we have $|B|=|S\cap B|=1$, say $B=S\cap B=\{z\}$. Then, we again find that z is an admissible vertex for $(x,y;A)$, which contradicts the assumption. This contradiction proves Lemma 4. ∎

Lemma 5 Let x be a vertex of a contraction-critically 5-connected graph G. Let A be a fragment with respect to $E(x)$ such that $|\bar{A}|\geq 2$, $|A|=2$. If there is neither an A-inner x^*-triangle nor an admissible vertex for $(x;A)$, then, $A\subseteq V_5$.

Proof. Let $A=\{u,u'\}$ and assume that either $u\notin V_5$ or $u'\notin V_5$. Let $S=N(A)=\{x,w,w',w'',w'''\}$. We may assume $u\in N(x)\cap A$. Since A is a fragment with respect to $E(x)$, we also assume that $w\in N(x)\cap S$.

Claim 5.1 $S-\{x\}\subseteq N(u')$.

Proof. If $u'\notin V_5$, then $N(u')=S\cup\{u\}$ and we are done. Hence assume $u'\in V_5$. If $u'x\in E(G)$, then we see that $G[\{x,u,u'\}]$ is an A-inner x^* -triangle, which violates the assumption. Hence $u'x\notin E(G)$, which implies the desired conclusion, $S-\{x\}\subseteq N(u')$. ■

Let B be a fragment with respect to xu and let $T=N(B)$.

Claim 5.2 (1) $u'\in T$ *and* (2) $|S\cap B|=|S\cap \bar{B}|=2$.

Proof. (1) By Claim 5.1, we see that $S-\{x\}\subseteq N(u')$, which implies $u'\in T$.

(2) Assume $|S\cap B|\leq 1$. Then $\bar{A}\cap B=\emptyset$ since $|S\cap B|<|A\cap T|$. If $S\cap B=\emptyset$, then $B=\emptyset$, which contradicts the choice of *B*. Hence assume $|S\cap B|=1$ and let $S\cap B=\{y\}$. Then we see that y is an admissible vertex for $(x;A)$, which contradicts the assumption. Hence $|S\cap B|\geq 2$. Similarly we see $|S\cap \bar{B}|\geq 2$. Then, since $S\cap T\neq\emptyset$, we have $|S\cap B|=|S\cap \bar{B}|=2$. ■

By Claim 5.2 (2), we may assume that $S\cap B=\{w,w'\}$ and $S\cap\bar{B}=\{w'',w'''\}$.

Claim 5.3 *If* $uw\in E(G)$, *then* $w\notin V_5$.

Proof. Assume that $uw\in E(G)$ and $w\in V_5$. Then, by Claim 5.1, we see that $u'w\in E(G)$. This implies w is an admissible vertex for $(x;A)$, which contradicts the assumption.

Claim 5.4 $u\in V_5$.

Proof. Assume $u\notin V_5$. Then $N(u)=S\cup\{u'\}$. Hence $uw\in E(G)$ and Claim 4.3 assures us that $w\notin V_5$. By Claim 5.1, we know that $u'w'\in E(G)$. Let C be a fragment with respect to $u'w'$ and let $R=N(C)$. Then, since $S\subseteq N(u)$, we see that $u\in R$, which implies $\{u,u'\}\subseteq T\cap R$.
■

Subclaim 5.4.1 $w\in R$.

Proof. Assume $w\notin R$. Without loss of generality we may assume that $w\in C$. Then, since $xw\in E(G)$, we observe that $x\in R\cup C$. Since $S\cap\bar{C}\neq\emptyset$ we see that $\{w'',w'''\}\cap\bar{C}\neq\emptyset$, which implies $(\bar{B}\cap\bar{C})\cap\{w'',w'''\}\neq\emptyset$ since $\{w'',w'''\}\subseteq\bar{B}$. Now we observe that $w\in B\cap C$ and

$(\bar{B}\cap\bar{C})\cap\{w'',w'''\}\neq\emptyset$, which implies that $|(R\cap B)\cup(R\cap T)\cup(C\cap T)|=5$. Hence $B\cap C$ is a fragment of G. Since $\{w'',w'''\}\subseteq\bar{B}$, $x\in T$ and $w'\in R$, we see that $N(\{u,u'\})\cap(B\cap C)=\{w\}$. Hence, applying Lemma 1 with the roles of A and S replaced by $B\cap C$ and $\{u,u'\}$, respectively, we see that $C\cap B=\{w\}$. This implies $w\in V_5$, which contradicts Claim 5.3. This contradiction proves Subclaim 5.4.1. ∎

Subclaim 5.4.2 (1) $x\in V_5$, *and* (2) $xu',xw'\in E(G)$.

Proof. (1) By Subclaim 5.4.1, we know that $\{w,w'\}\subseteq S\cap R$, which implies either $|S\cap C|=1$ or $|S\cap\bar{C}|=1$. Without loss of generality we may assume that $|S\cap C|=1$, say $S\cap C=\{z\}$. Then $z\in\{x,w'',w'''\}$. Since $|S\cap C|<|A\cap R|$, Lemma 2 (2) assures us that $\bar{A}\cap C=\emptyset$, which implies $C=S\cap C=\{z\}$. Hence $z\in V_5$ and $zw\in E(G)$. Since $ww'',ww'''\notin$ $\notin E(G)$, we see that $z=x$ and $x\in V_5$.

(2) Since $N(x)=R$, we observe that $xu',xw'\in E(G)$. ∎

Subclaim 5.4.3 $ww'\in E(G)$.

Proof. Since $|A\cap T|=|S\cap B|=2$, we see that $|(\bar{A}\cap T)\cup(S\cap T)\cup(S\cap B)|=5$. Let $N(x)=\{u,u',w,w',v\}$. Since $N(x)\cap\bar{A}\neq\emptyset$ and $\{u,u',w,w'\}\subseteq(A\cap T)\cup(S\cap B)$, we observe that $v\in\bar{A}\cap\bar{B}$, which implies $N(x)\cap(\bar{A}\cap B)=\emptyset$. Since $|(\bar{A}\cap T)\cup(S\cap T)\cup(S\cap B)|=5$ and $N(x)\cap(\bar{A}\cap B)=\emptyset$, we see that $\bar{A}\cap B=\emptyset$, which implies $B=S\cap B=\{w,w'\}$. Since $w\notin V_5$ and $B=\{w,w'\}$, we have $ww'\in E(G)$. ∎

We proceed with the proof of Claim 5.4. Now we observe that $G[N(x)-\{v\}]\cong K_4$, which implies xv is contractible. This contradicts that G is contraction-critically 5-connected and Claim 5.4 is proved. ∎

By Claim 5.4, we have $u\in V_5$. Hence $u'\notin V_5$. But, in this situation, we see that $G[\{x,u,u'\}]$ is an A-inner x^* -triangle, which contradicts the assumption. This contradiction proves Lemma 5. ∎

Recall that an vertex x is said to be insufficient on a fragment A if (1) there is no A-inner x^* -triangle and (2) $N(u)\cap N(u')\cap A\cap V_5=\emptyset$ for any $u,u'\in N(x)\cap A\cap V_5$.

The following Lemma 6 says that "x is insufficient on A" is an sufficient condition for the existence of an admissible vertex for $(x;A)$.

Lemma 6 *Let x be a vertex of a contraction-critically* 5*-connected graph G. Let A be a fragment*

such that $x \in N(A)$, $|\bar{A}| \geq 2$ ***and*** $|A| \geq 3$. ***If x is insufficient on A, then there is an admissible vertex for*** $(x; A)$.

Proof. We prove Lemma 6 by the induction on $|N(x) \cap A|$. If $|N(x) \cap A| = 1$, then Lemma 3 assures us that the desired conclusion holds. Assume $|N(x) \cap A| \geq 2$ and also assume that there is no admissible vertex for $(x; A)$. Choose $y \in N(x) \cap A$ so that $\deg_G(y)$ to be as small as possible. Since there is no admissible vertex for $(x, y; A)$, Lemma 5 assures us that there is a fragment A' with respect to xy such that $A' \subsetneq A$.

Claim 6.1 $|A'| = 2$.

Proof. At first assume $|A'| = 1$, say $A' = \{u\}$. Then $u \in V_5$, $\{x, y\} \subseteq N(u)$ and $A = \{y, u\}$. In this situation, we observe that $G[\{x, y, u\}]$ is an A-inner x^*-triangle, which violates the fact that x is insufficient on A.

Next assume $|A'| \geq 3$. Then $|A'| \geq 3$ and $|\bar{A}'| > |\bar{A}| \geq 2$. Since x is insufficient on A and $A' \subsetneq A$, x is also insufficient on A'. Since $y \in N(x) \cap A$ and $y \notin N(x) \cap A'$, we see that $|N(x) \cap A'| < |N(x) \cap A|$. Hence, applying the induction hypothesis to A', we see that there is an admissible vertex z for $(x; A')$. Since $A' \subsetneq A$, $N(A') \subseteq S \cup A$, which implies $z \in S \cup A$. We show $z \in S$. Assume that $z \in A$. Since z is an admissible vertex for $(x; A')$, there is a vertex $u \in N(x) \cap N(z) \cap A'$. Then, since $z \in A \cap V_5$ and $u \in A' \subsetneq A$, we observe that $G[\{x, z, u\}]$ is an A-inner x^*-triangle, which violates the fact that x is insufficient on A. Now it is shown that $z \in S$, which implies that z is an admissible vertex z for $(x; A)$. This contradicts the assumption and Claim 6.1 is proved. ∎

By Claim 6.1 we know $|A'| = 2$, say $A' = \{u, u'\}$. We may assume that $xu \in E(G)$. Since $A' \subsetneq A$ and there is no A-inner x^*-triangle, we see there is no A'-inner x^*-triangle. Assume that there is an admissible vertex z for $(x; A')$. Then $z \in V_5$ and $N(x) \cap N(z) \cap A' \neq \emptyset$. If $z \in A$, then we find an A-inner x^*-triangle, which contradicts the assumption. Hence $z \in S$ and z is an admissible vertex for $(x; A)$, which again contradicts the assumption. It is shown that there is no admissible vertex for $(x; A')$. Hence, there is neither an A'-inner x^*-triangle nor an admissible vertex for $(x; A')$. Thus Lemma 4 assures us that $u, u' \in V_5$. Recall that we choose y so that $\deg_G(y)$ to be as small as possible. Hence, we see that $y \in V_5$ since $u \in N(x) \cap A \cap V_5$. Since there is no A-inner x^*-triangle and $y, u \in N(x) \cap A \cap V_5$, we see that $yu \notin E(G)$, which implies $uu' \in E(G)$ since $A' = \{u, u'\}$. If $xu' \in E(G)$, then $G[\{x, u, u'\}]$ is an A-inner x^*-triangle, which contradicts the assumption. Hence $xu' \notin E(G)$, which implies $yu' \in E(G)$. Now we observe that $y, u \in N(x) \cap A \cap V_5$ and $u' \in N(y) \cap N(u) \cap A \cap V_5$, which contradicts the assumption that x is insufficient on A. This contradiction completes the proof of Lemma 6. ∎

We note that, in the definition of 'insufficient', the condition "(2) $N(u) \cap N(u') \cap A \cap V_5 = \emptyset$ for any $u, u' \in N(x) \cap A \cap V_5$" is necessary. There is a contraction-critically 5-connected graph G which has a vertex x and a fragment A such that $x \in N(A)$, $|\bar{A}| \geq 2$ and $|A| \geq 3$ and G has neither an admissible vertex for $(x; A)$ nor an A-inner x^*-triangle.

By the definition, if $N(x) \cap A \cap V_5 = \emptyset$, then x is insufficient on A. Hence, the following is an immediate corollary of Lemma 6.

Corollary 7 ([3] Lemma 6) *Let G be a contraction-critically 5-connected graph G and let A be a fragment of G with $|\bar{A}| \geq 2$ and $|A| \geq 3$. Let $x \in N(A)$. If $N(x) \cap A \cap V_5 = \emptyset$, then there is an admissible vertex for $(x; A)$.* ∎

Lemma 8 *Let x be a vertex of a contraction-critically 5-connected graph G. Let A be a fragment such that $x \in N(A)$, $|\bar{A}| \geq 2$ and $|A| \geq 3$. Suppose $|N(x) \cap A| = 1$ and $N(x) \cap A \cap V_5 = \emptyset$. Then,*

(1) *there is a strongly admissible vertex z for $(x; A)$,*

(2) *if $(N(z) \cap N(A) - \{x\}) \cap (V_5 - V_5^{(2)}) = \emptyset$, then z is a hyper admissible vertex for $(x; A)$.*

Proof. Let $S = N(A)$ and let $N(x) \cap A = \{y\}$. Note that $y \notin V_5$ since $N(x) \cap A \cap V_5 = \emptyset$. By Lemma 3, there is an admissible vertex z for $(x, y; A)$. Let $B = \{z\}$ and let $T = N(y) = N(B)$.

We show (1). Assume z is not strongly admissible, that is $|N(z) \cap \bar{A}| \geq 2$. Then, since $z \in V_5$, we see that $|N(z) \cap \bar{A}| = |N(z) \cap A| = 2$, $S \cap T = \{x\}$ and $|S \cap \bar{B}| = 3$. Let $A \cap T = \{y, u\}$ and let $S \cap \bar{B} = \{w, w', w''\}$. Furthermore, let $A' = A - \{y\}$ and $S' = N(A') = (S - \{x\}) \cup \{y\} = \{z, y, w, w', w''\}$. Since $N(x) \cap A = \{y\}$, we observe that A' is a fragment of G such that $|A'| = |A - \{y\}| \geq 2$ and $|\bar{A}'| = |\bar{A} \cup \{x\}| \geq 3$. Then, since $N(z) \cap S' = \{y\}$ and $y \notin V_5$, we observe that $N(z) \cap S' \cap V_5 = \emptyset$, which implies that there is no admissible vertex for $(z; A')$. If $|A'| \geq 3$, then Lemma 3 assures us the existence of an admissible vertex for $(z; A')$, which contradicts the previous assertion. Hence we have $|A'| = 2$, say $A' = \{u, u'\}$. Then $u' \in A' \cap \bar{B}$ and $N(u') = \{u, y, w, w', w''\}$. Moreover we observe that $N(u) \subseteq \{y, z, u', w, w', w''\}$ and $N(y) \subseteq \{x, z, u, u', w, w', w''\}$. Since $y \notin V_5$, we see that $|N(y) \cap \{w, w', w''\}| \geq 2$. Without loss of generality, we may assume that $\{w, w'\} \subseteq N(y)$. Let B' be a fragment with respect to zu and let $T' = N(B')$. Since $N(z) \cap N(u) \subseteq \{y\}$ and $y \notin V_5$, we see that $N(z) \cap N(u) \cap V_5 = \emptyset$, which implies that neither B' nor $\bar{B}'$ is trivial, and hence $|B'| \geq 2$ and $|\bar{B}'| \geq 2$. Since $S' - \{z\} \subseteq N(u')$, we see that $u' \in T'$.

Claim 8.1 $y \in T'$.

Proof. Assume $y \notin T'$. Without loss of generality, we may assume that $y \in B'$. Then, since $\{w, w'\} \subseteq N(y)$, $\{w, w'\} \subseteq T' \cup B'$. Hence, we observe that $N(\{u, u'\}) \cap \bar{B}' = \{w''\}$. Then, assures us that $\bar{B}' = \{w''\}$, which contradicts the previous observation that $|\bar{B}'| \geq 2$. This contradiction proves Claim 8.1. ∎

By Claim 8.1, we see that $\{y, z, u, u'\} \subseteq T'$, which implies $N(u) \cap (B' \cup \bar{B}') \subseteq \{w, w', w''\}$ since $N(u) \subseteq \{y, z, u', w, w', w''\}$. We also observe that $N(u') \cap (B' \cup \bar{B}') \subseteq \{w, w', w''\}$ since $N(u') = \{u, y, w, w', w''\}$. Since neither $N(u) \cap B' = \emptyset$ nor $N(u) \cap \bar{B}' = \emptyset$, we have either $|B' \cap \{w, w', w''\}| = 1$ or $|\bar{B}' \cap \{w, w', w''\}| = 1$. Without loss of generality, we may assume that $|B' \cap \{w, w', w''\}| = 1$, say $B' \cap \{w, w', w''\} = \{\tilde{w}\}$. Then we see that $N(\{u, u'\}) \cap B' = \{\tilde{w}\}$ and applying Lemma 1 with the roles of A and S replaced by B' and $\{u, u'\}$, respectively, we see that $B' = \{\tilde{w}\}$, which contradicts the previous observation that $|B'| \geq 2$. This contradiction proves that z is a strongly admissible vertex for $(x, y; A)$ and (1) is shown.

Next we show (2). Assume z is not a hyper admissible vertex for $(x; A)$. We show $(N(z) \cap S - \{x\}) \cap (V_5 - V_5^{(2)}) \neq \emptyset$. Since z is strongly admissible and not hyper admissible, we see that $|N(z) \cap A| = 2$, $|N(z) \cap S| = 2$, $|N(z) \cap \bar{A}| = 1$ and $|S \cap \bar{B}| = 2$. Let $N(z) \cap A = \{y, u\}$, $N(z) \cap S = \{x, w\}$, $N(z) \cap \bar{A} = \{v\}$ and $S \cap \bar{B} = \{w', w''\}$. Let $A' = A - \{y\}$ and $S' = N(A') = (S - \{x\}) \cup \{y\} = \{z, y, w, w', w''\}$. Then A' is a fragment of G such that $|A'| \geq 2$ and $|\bar{A}'| = |\bar{A} \cup \{x\}| \geq 3$. Note that $N(z) \cap A' = \{u\}$.

Claim 8.2 *w is an admissible vertex for* $(z, u; A')$.

Proof. At first we consider the case that $|A'| \geq 3$. In this case we have $|A'| \geq 3$, $|\bar{A}'| \geq 3$ and $N(z) \cap A' = \{u\}$. Thus Lemma 3 assures us the existence of an admissible vertex for $(z, u; A')$. Since $N(z) \cap S' = \{y, w\}$ and $y \notin V_5$, we observe that w is an admissible vertex for $(z, u; A')$.

Next we consider the case that $|A'| = 2$, say $A' = \{u, u'\}$. Since $A' \cap B = \emptyset$ and $A' \cap T = \{u\}$, we see that $u' \in A' \cap \bar{B}$ and $N(u') = \{y, u, w, w', w''\}$. Since $N(y) \subseteq S \cup A$ and $A = \{y, u, u'\}$, the fact $y \in V_{\geq 6}$ implies $|N(y) \cap \{w, w', w''\}| \geq 2$. Let B' be a fragment with respect to zu and let $T' = N(B')$. Since $S' - \{z\} \subseteq N(u')$, we observe that $u' \in T'$, which implies $A' \cap T' = \{u, u'\}$ and $A' \cap B' = A' \cap \bar{B}' = \emptyset$. Since $A' \cap B' = A' \cap \bar{B}' = \emptyset$, we see that neither $S' \cap B' = \emptyset$ nor $S' \cap \bar{B}' = \emptyset$. We show that either $|S' \cap B'| = 1$ or $|S' \cap \bar{B}'| = 1$. If $y \in S' \cap T'$, then $|S' \cap T'| \geq 2$, which implies either $|S' \cap B'| = 1$ or $|S' \cap \bar{B}'| = 1$. Hence assume $y \notin S' \cap T'$. If $y \in S' \cap \bar{B}'$, then, the fact that $|N(y) \cap \{w, w', w''\}| \geq 2$ assures us that $|S' \cap B'| = 1$. Similarly, if $y \in S' \cap B'$, then we have $|S' \cap \bar{B}'| = 1$. Now it is shown that either $|S' \cap B'| = 1$ or $|S' \cap \bar{B}'| = 1$.

Without loss of generality, we may assume that $|S' \cap B'| = 1$, say $S' \cap B' = \{\tilde{w}\}$. Then, since $|S' \cap B'| < |A' \cap T'|$, we observe that $\bar{A}' \cap B' = \emptyset$ and $B' = S' \cap B' = \{\tilde{w}\}$. Hence we know that $\tilde{w} \in V_5$ and $\tilde{w}z \in E(G)$. Since $N(z) \cap S' = \{y, w\}$ and $y \notin V_5$, we see that $\tilde{w} = w$, which implies the desired conclusion that w is an admissible vertex for $(z, u; A')$. ∎

If $w \notin V_5^{(2)}$, then $w \in (N(y) \cap S - \{x\}) \cap (V_5 - V_5^{(2)})$ and we are done. Hence assume $w \in V_5^{(2)}$.

Claim 8.3 *If* $N(w) \cap \bar{A} \cap V_5 = \emptyset$, *then* $|\bar{A}| \geq 3$.

Proof. Assume $N(w) \cap \bar{A} \cap V_5 = \emptyset$. Since $\bar{A} \cap B = \emptyset$, $\bar{A} \cap T = \{v\}$ and $|\bar{A}| \geq 2$, we observe $\bar{A} \cap \bar{B} \neq \emptyset$, which implies $\bar{A} \cap \bar{B}$ is a fragment of G since $|(S \cap \bar{B}) \cup (S \cap T) \cup (\bar{A} \cap T)| = 5$. Hence $N(w) \cap (\bar{A} \cap \bar{B}) \neq \emptyset$, say $v' \in N(w) \cap (\bar{A} \cap \bar{B})$. Then, since $N(w) \cap \bar{A} \cap V_5 = \emptyset$, we see that $v' \notin V_5$, which implies $|\bar{A} \cap \bar{B}| \geq 2$. This implies the desired conclusion $|\bar{A}| = |\bar{A} \cap T| + |\bar{A} \cap \bar{B}| \geq 3$. ∎

Claim 8.4 $N(w) \cap A \cap V_5 = \emptyset$.

Proof. Assume $N(w) \cap A \cap V_5 \neq \emptyset$. Then, since $z \in N(w) \cap V_5$ and $w \in V_5^{(2)}$, we see that $N(w) \cap \bar{A} \cap V_5 = \emptyset$. Hence Claim 8.3 assures us that $|\bar{A}| \geq 3$. Since $|\bar{A}|, |A| \geq 3$ and $N(w) \cap \bar{A} \cap V_5 = \emptyset$, applying Corollary 7, we see that there is an admissible vertex for $(w; \bar{A})$. Since $z \in N(w) \cap V_5$, $N(w) \cap A \cap V_5 \neq \emptyset$ and $w \in V_5^{(2)}$, we observe that $N(w) \cap S \cap V_5 = \{z\}$. Since $|N(z) \cap \bar{A}| = 1$, z is not an admissible vertex for $(w; \bar{A})$, which implies that there is no admissible vertex for $(w; \bar{A})$. This contradicts the previous assertion and this contradiction proves Claim 8.4. ∎

Claim 8.5 $|A'| \geq 3$.

Proof. Since $A' \cap B = \emptyset$, $A' \cap T = \{u\}$ and $|A'| \geq 2$, we observe that $A' \cap \bar{B} \neq \emptyset$, which implies that $A' \cap \bar{B}$ is a fragment of G since $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)| = 5$, which implies $|A' \cap \bar{B}| \geq 2$. This implies the desired conclusion that $|A'| = |A' \cap T| + |A' \cap \bar{B}| \geq 3$. ∎

We proceed with the proof of Lemma 8 (2).

Since $|\bar{A}'|, |A'| \geq 3$ and $N(w) \cap A' \cap V_5 = \emptyset$, applying Corollary 7, we see that there is an admissible vertex $\tilde{w}$ for $(w; A')$. Since $|N(z) \cap A'| = 1$, z is not an admissible vertex for $(w; A')$, which implies $\tilde{w} \neq z$. Then, since $w \in V_5^{(2)}$, we observe that $N(w) \cap V_5 = \{z, \tilde{w}\}$, which implies that $N(w) \cap \bar{A}' \cap V_5 = \emptyset$. Since $\bar{A} = \bar{A}' - \{x\}$, $N(w) \cap \bar{A}' \cap V_5 = \emptyset$ implies $N(w) \cap \bar{A} \cap V_5 = \emptyset$. Now we have $N(w) \cap \bar{A} \cap V_5 = \emptyset$ and Claim 8.3 assures us that $|\bar{A}| \geq 3$. Since $|\bar{A}|, |A| \geq 3$, $|N(w) \cap \bar{A}| = 1$

and $N(w)\cap\bar{A}\cap V_5=\emptyset$, applying (1), we see that there is a strongly admissible vertex for $(w;\bar{A})$. However, since $N(w)\cap S\cap V_5=\{z,\tilde{w}\}$, $|N(z)\cap A|\geq 2$ and $|N(\tilde{w})\cap A|\geq 2$, we see that there is no strongly admissible vertex for $(w;\bar{A})$, which violates the previous assertion. This contradiction proves (2) and the proof of Lemma 8 is completed. ∎

4 The proof of Theorem 1

In this section we give a proof of Theorem 1.

Let G be a 5-connected graph. Let A be a fragment of G and let $S=N(A)$. Let Ad $(Y;A)$ denote the set of admissible vertices for $(Y;A)$. We demote $\hat{S}_A$ the set of vertices y of S such that $Ad(y;A)\neq\emptyset$ and let $\tilde{S}_A=\cup_{y\in\hat{S}_A}Ad(y;A)$. Using these notation, we can rewrite Theorem 1 as the following.

Theorem 1 *Let G be a contraction-critically 5-connected graph. Let A be a connected fragment of G with $|A|=2$, say $A=\{x_1,x_2\}$ and let $S=N(A)$.*

(1) *If $A\cap V_6\neq\emptyset$, then $|\hat{S}_A|\geq 4$.*

(2) *If $A\cap V_6\neq\emptyset$, $|\tilde{S}_A|\geq 3$.*

(3) *If $A\cap V_6=\emptyset$, then $\hat{S}_A\cap(S-N(x_1)\cap N(x_2))\neq\emptyset$.*

We prove Theorem 1 using the notation $\hat{S}_A$ and $\tilde{S}_A$. Let $S=\{y_1,y_2,y_3,y_4,y_5\}$. Without loss of generality we may assume that $\deg_G(x_1)\geq\deg_G(x_2)$. Hence, if $A\cap V_6\neq\emptyset$, then $x_1\in V_6$ and $S\subseteq N(x_1)$.

(1) Assume $A\cap V_6\neq\emptyset$ and $|\hat{S}_A|\leq 3$. Then $|S-\hat{S}_A|\geq 2$, say $y_1,y_2\in S-\hat{S}_A$.

We show that there is a fragment B_i such that $\{x_1,x_2,y_i\}\subseteq N(B_i)$ for $i=1,2$. Let $i\in\{1,2\}$. If $x_2y_i\in E(G)$, then let B_i be a fragment with respect to x_2y_i. Then, since $S\subseteq N(x_1)$, we observe that $\{x_1,x_2,y_i\}\subseteq N(B_i)$. If $x_2y_i\notin E(G)$, then let B_i be a fragment with respect to x_1y_i. Then, since $S-\{y_i\}\subseteq N(x_2)$, we again observe that $\{x_1,x_2,y_i\}\subseteq N(B_i)$. Now the existence of a fragment B_i such that $\{x_1,x_2,y_i\}\subseteq N(B_i)$ is shown.

Let B_i be a fragment such that $\{x_1,x_2,y_i\}\subseteq N(B_i)$ and let $T_i=N(B_i)$ for $i=1,2$. We show that $|S\cap B_1|\geq 2$. Suppose $|S\cap B_1|\leq 1$. Then, since $|S\cap B_1|<|A\cap T_1|$, Lemma 2 (2) assures us that $\bar{A}\cap B_1=\emptyset$, which implies $B_1=S\cap B_1$, say $B_1=S\cap B_1=\{y\}$. Then we observe $y\in V_5$ and $\{y_1\}\cup A\subseteq N(y)$, which implies that y is an admissible vertex for $(y_1;A)$. This contradicts the fact that $y_1\in S-\hat{S}_A$ and it is shown that $|S\cap B_1|\geq 2$.

By the similar arguments, we can show that $|S\cap\bar{B}_1|,|S\cap B_2|,|S\cap\bar{B}_2|\geq 2$. Thus we have

$|S\cap B_i|=|S\cap \bar{B}_i|=2$ for $i=1,2$. Without loss of generality we may assume that $S\cap B_1=\{y_2,y_3\}$ and $S\cap \bar{B}_1=\{y_4,y_5\}$. Say $S\cap B_2=\{y_1,y_j\}$ and $S\cap \bar{B}_2=\{y_3,y_4,y_5\}-\{y_j\}$. Then we observe that $y_1\in T_1\cap B_2$ and $y_2\in T_2\cap B_1$.

We show $j\neq 3$. Suppose $j=3$. Then $y_3\in B_1\cap B_2$ and $y_4,y_5\in \bar{B}_1\cap \bar{B}_2$. Since neither $B_1\cap B_2$ nor $\bar{B}_1\cap \bar{B}_2$ is empty, we see that $B_1\cap B_2$ is a fragment of G. Since $\{x_1,x_2\}\subseteq N(B_1\cap B_2)$ and $N(\{x_1,x_2\})\cap(B_1\cap B_2)=\{y_3\}$, applying Lemma 1 with the roles of A and S replaced by $B_1\cap B_2$ and $\{x_1,x_2\}$, respectively, we see that $B_1\cap B_2=\{y_3\}$, which implies $y_3\in V_5$ and $\{y_1\}\cup A\subseteq N(y_3)$. Hence $y_3\in Ad(y_1;A)$, which contradicts the choice of y_1. This contradiction shows $j\neq 3$, say $j=4$.

In this situation, we observe that $y_3\in B_1\cap \bar{B}_2$, $y_4\in \bar{B}_1\cap B_2$ and $y_5\in \bar{B}_1\cap \bar{B}_2$. Since neither $\bar{B}_1\cap B_2$ nor $B_1\cap \bar{B}_2$ is empty, we see that $\bar{B}_1\cap B_2$ is a fragment of G. Since $\{x_1,x_2\}\subseteq N(\bar{B}_1\cap B_2)$ and $N(\{x_1,x_2\})\cap(\bar{B}_1\cap B_2)=\{y_4\}$, applying Lemma 1 with the roles of A and S replaced by $\bar{B}_1\cap B_2$ and $\{x_1,x_2\}$, respectively, we see that $\bar{B}_1\cap B_2=\{y_4\}$, which implies $y_4\in V_5$ and $\{y_1\}\cup A\subseteq N(y_4)$. Hence $y_4\in Ad(y_1;A)$, which again contradicts the choice of y_1. This contradiction shows that $|\hat{S}_A|\geq 4$ and (1) is proved.

(2) Assume $A\cap V_6\neq\emptyset$ and $|\tilde{S}_A|\leq 2$. Since $\hat{S}_A\neq\emptyset\geq 4$, we see that $\tilde{S}_A\neq\emptyset$, say $y\in\tilde{S}_A$. Since $y\in V_5$, $A\subseteq N(y)$ and $N(y)\cap\bar{A}\neq\emptyset$, we see that $|N(y)\cap S|\leq 2$. Since $|\hat{S}_A|\geq 4$ and $|N(y)\cap S|\leq 2$ for $y\in\tilde{S}_A$, we see that $|\tilde{S}_A|=2$, say $\tilde{S}_A=\{y_1,y_2\}$. Since $|\hat{S}_A|\geq 4$, we see that either $y_1\in\hat{S}_A$ or $y_2\in\hat{S}_A$, which implies $y_1y_2\in E(G)$ and $\{y_1,y_2\}\subseteq\hat{S}_A$. Since $|N(y)\cap S|\leq 2$ for $y\in\tilde{S}_A$ and $y_1y_2\in E(G)$, we see that $|N(\tilde{S}_A)\cap S|\leq 2$, which implies $|\hat{S}_A|\leq 4$ and $S-\hat{S}_A\neq\emptyset$, say $y_j\in S-\hat{S}_A$.

By the same arguments in the proof of (1), we see there is a fragment B with $\{x_1,x_2,y_j\}\subseteq N(B)$ and we also have $|S\cap B|=|S\cap\bar{B}|=2$. Let $T=N(B)$. Since $y_1y_2\in E(G)$, we may assume that $S\cap B=\{y_1,y_2\}$. Since $E_G(S\cap B,S\cap\bar{B})=\emptyset$ and $S\cap B=\tilde{S}_A$, we see that $N(\tilde{S}_A)\cap(S\cap\bar{B})=\emptyset$, which implies $\hat{S}_A\cap(S\cap\bar{B})=\emptyset$ and $|\hat{S}_A|=|\tilde{S}_A|=2$. This contradicts (1) and this contradiction shows $|\tilde{S}_A|\geq 3$. Now (2) is proved.

(3) Assume $\hat{S}_A\cap(S-N(x_1)\cap N(x_2))=\emptyset$. Since $|A|=2$ and $A\cap V_6=\emptyset$, we observe that $A\subseteq V_5$, which implies $|N(x_1)\cap N(x_2)|=3$, say $N(x_1)\cap N(x_2)=\{y_3,y_4,y_5\}$ and $N(x_i)-\{x_{3-i},y_3,y_4,y_5\}=\{y_i\}$ for $i=1,2$. Then $S-N(x_1)\cap N(x_2)=\{y_1,y_2\}$ and by the assumption, we observe that $\hat{S}_A\cap\{y_1,y_2\}=\emptyset$. Since $\hat{S}_A\cap\{y_1,y_2\}=\emptyset$, we observe that $Ad(y_i;A)=\emptyset$ for $i=1,2$. Let B_i be a fragment with respect to x_iy_i and let $T_i=N(B_i)$ for $i=1,2$.

We show that $x_2\in T_1$. Suppose $x_2\notin T_1$, say $x_2\in\bar{B}_1$. Then since $N(x_2)=A\cup S-\{x_2,y_1\}$, we observe that $A\cup S\subseteq\bar{B}_1\cup T_1$, which implies $N(x_1)\cap B_1=\emptyset$. This contradicts the choice of B_1 and it is shown that $x_2\in T_1$.

Similarly we have $x_1 \in T_2$. Now we know that $\{x_1, x_2, y_1, y_2\} \subseteq T_1 \cup T_2$. Since $\{x_1, x_2, y_1, y_2\} \subseteq T_1 \cup T_2$ and neither $N(x_1) \cap B_1$ nor $N(x_1) \cap \bar{B}_1$ is empty, we see neither $B_1 \cap \{y_3, y_4, y_5\}$ nor $\bar{B}_1 \cap \{y_3, y_4, y_5\}$ is empty, which implies either $|B_1 \cap \{y_3, y_4, y_5\}| = 1$ or $|\bar{B}_1 \cap \{y_3, y_4, y_5\}| = 1$, say $|B_1 \cap \{y_3, y_4, y_5\}| = 1$ and let $B_1 \cap \{y_3, y_4, y_5\} = \{y_3\}$.

We show that $y_2 \in B_1$. Suppose $y_2 \notin B_1$. Then, since $N(\{x_1, x_2\}) \cap B_1 = \{y_3\}$, applying Lemma 1 with the roles of A and S replaced by B_1 and $\{x_1, x_2\}$, respectively, we see that $B_1 = \{y_3\}$, which implies $y_3 \in V_5$ and $\{y_1\} \cup A \subseteq N(y_3)$. Hence we have $y_3 \in Ad(y_1; B_1)$, which contradicts the assumption that $\hat{S}_A \cap \{y_1, y_2\} = \emptyset$. This contradiction shows that $y_2 \in B_1$.

We show that $\{y_4, y_5\} \subseteq \bar{B}_1$. Suppose $y_5 \notin \bar{B}_1$. Then, since $N(\{x_1, x_2\}) \cap \bar{B}_1 = \{y_4\}$, applying Lemma 1 with the roles of A and S replaced by $\bar{B}_1$ and $\{x_1, x_2\}$, respectively, we see that $\bar{B}_1 = \{y_4\}$, which implies $y_4 \in Ad(y_1; A)$. This contradicts the assumption that $\hat{S}_A \cap \{y_1, y_2\} = \emptyset$ and it is shown that $y_5 \in \bar{B}_1$.

By symmetry we have $y_4 \in \bar{B}_1$. By the similar argument, we know that there is an integer $j \in \{3, 4, 5\}$ such that $\{y_1, y_j\} \subseteq B_2$ and $\{y_3, y_4, y_5\} - \{y_j\} \subseteq \bar{B}_2$. In this situation we observe that $y_1 \in T_1 \cap B_2$ and $y_2 \in T_2 \cap B_1$.

We show $j \neq 3$. Suppose $j = 3$. Then $y_3 \in B_1 \cap B_2$ and $y_4, y_5 \in \bar{B}_1 \cap \bar{B}_2$. Since neither $B_1 \cap B_2$ nor $\bar{B}_1 \cap \bar{B}_2$ is empty, Lemma 2 (1) assures us that $B_1 \cap B_2$ is a fragment of G. Since $\{x_1, x_2\} \subseteq N(B_1 \cap B_2)$ and $N(\{x_1, x_2\}) \cap (B_1 \cap B_2) = \{y_3\}$, applying Lemma 1, with the roles of A and S replaced by $B_1 \cap B_2$ and $\{x_1, x_2\}$, respectively, we see that $B_1 \cap B_2 = \{y_3\}$, which implies $y_3 \in V_5$ and $\{y_1\} \cup A \subseteq N(y_3)$. Hence $y_3 \in Ad(y_1; A)$, which contradicts the assumption that $\hat{S}_A \cap \{y_1, y_2\} = \emptyset$ and it is shown that $j \neq 3$, say $j = 4$.

Then $y_3 \in B_1 \cap \bar{B}_2$, $y_4 \in \bar{B}_1 \cap B_2$ and $y_5 \in \bar{B}_1 \cap \bar{B}_2$. Since neither $\bar{B}_1 \cap B_2$ nor $B_1 \cap \bar{B}_2$ is empty, we see that $\bar{B}_1 \cap B_2$ is a fragment of G. Since $\{x_1, x_2\} \subseteq N(\bar{B}_1 \cap B_2)$ and $N(\{x_1, x_2\}) \cap (\bar{B}_1 \cap B_2) = \{y_4\}$, applying Lemma 1 with the roles of A and S replaced by $\bar{B}_1 \cap B_2$ and $\{x_1, x_2\}$, respectively, we see that $\bar{B}_1 \cap B_2 = \{y_4\}$, which implies $y_4 \in V_5$ and $\{y_1\} \cup A \subseteq N(y_4)$. Hence $y_4 \in Ad(y_1; A)$, which contradicts the assumption that $\hat{S}_A \cap \{y_1, y_2\} = \emptyset$. This contradiction shows that $\hat{S}_A \cap (S - N(x_1) \cap N(x_2)) \neq \emptyset$. Now (3) is proved and the proof of Theorem 1 is completed. ∎

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References

[1] K. Ando, A local structure theorem on 5-connected graphs, *J. Graph Theory* **60** (2009), 99-129.
[2] K. Ando, Y. Egawa, and M. Kriesell The average degree of minimally contraction-critically 5-connected graphs, *J. Graph Theory* **75** (2014), no. 4, 331-354.
[3] K. Ando and T. Iwase, The number of vertices of degree 5 in a contraction-critically 5-connected graph, *Discrete Math.* **311** (2011), 1925-1939.
[4] K. Ando, A. Kaneko and K. Kawarabayashi, Vertices of degree 5 in a contraction critically 5-connected graphs, *Graphs Combin.* **21** (2005), 27-37.
[5] K. Ando, A. Kaneko, K. Kawarabayashi and K. Yoshimoto, Contractible edges and bowties in a k-connected graph, *Ars Combin.* **64** (2002), 239-247.
[6] K. Ando and K. Kawarabayashi, Some forbidden subgraph conditions for a graph to have a k-contractible edge, *Discrete Math.* **267** (2003), 3-11.
[7] Y. Egawa, Contractible edges in n-connected graphs with minimum degree greater that or equal to $\left[\frac{5n}{2}\right]$, *Graphs Combin.* **7** (1991), 15-21.
[8] M. Fontet, Graphes 4-essentiels, *C. R. Acad. Sci. Paris* **287** (1978), 289-290.
[9] M. Kriesell, A survey on contractible edges in graphs of a prescribed vertex connectivity, *Graphs Combin.* **18** (2002), 1-30.
[10] N. Martinov, Uncontractible 4-connected graphs, *J. Graph Theory* **6** (1982), 343-344.
[11] J. Su, Vertices of degree 5 in contraction critical 5-connected graphs, *J. Guangxi Norm. Univ.* **3** (1997), 12-16 (in Chinese).
[12] C. Thomassen, Non-separating cycles in k-connected graphs, *J. Graph Theory* **5** (1981), 351-354.
[13] W. T. Tutte, A theory of 3-connected graphs, *Indag. Math.* **23** (1961), 441--455.