Some Models in Stochastic Processes

Dedicated to Professor Hideo Ōsawa

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abstract

In this paper we review the four models in Stochastic Processes. First one described by *integro-partial differential equation or integro-differential equation* is found in the process which has two phases, the "inflow" and the "outflow", and the switchover of which is controlled by some storage level. Second we find it as *renewal equation* in the same process. We deal with the third one in the Ruin Process using recursive equation. Fourth one is the Queue with *Arrival Acceptance Window*.

1 Storage process

First of all we investigate the storage process with upper boundary which has two phases, called as the inflow and outflow phases, and the switch over of these phases is controlled by a certain storage level.

In the inflow phase the storage is increasing and in the outflow phase the storage is decreasing. Assume that the storage increases or decreases at each rate dependent on the present phase and level and that the inflow has two different increasing rates. In both phases the large scale demand for the system may occur according to the Poisson process. We present the analytical solution for the steady-state probabilities of storage levels and the ruin probability incurred the first epoch at which the storage level drops down below the zero level.

For this process Doi and Ōsawa [1] have studied numerically on the steady-state probabilities of storage levels and Doi [2, 5] has got the mean ruin time. For the simple process with increasing rate, Doi [3, 6] has studied on the mean ruin time and Doi, Nagai and Ōsawa [4] have got the ruin probability.

Let X_t be the storage level at time t. Assume that it has boundaries L and zero, that is, $0 \leq X_t \leq L$. We define a time interval in which X_t is increasing as an inflow phase, and during this

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phase X_t has inflow rates $\alpha_2(x)$ given that $X_t = x$ for $l \le x < L$ and $\alpha_1(x)$ given that $X_t = x$ for $0 < x < l$. We also refer to a time interval in which X_t is decreasing as an outflow phase having a rate $\alpha_0(x)$ $(l < x < L)$. Once X_t reaches the upper bound L , it remains at the level in a certain period whose length is exponentially distributed with parameter ν_L . Immediately after this period, the phase changes to the outflow one and X_t is controlled according to the outflow rate $\alpha_0(x)$ given that $X_t = x$ for $l < x < L$.

Throughout these phases, the large scale demand for the system may occur according to the Poisson process, that is, the inter-occurrence time has an exponential distribution with parameter λ . Let the amount of each demand has a distribution function $F(x)$ with the density function $f(x)$ having the finite mean.

There are two cases for the switch over from the outflow phase to the inflow one. First, if X_t decreases to the level l continuously, the phase instantaneously changes to the inflow one with rate $\alpha_2(x)$. Second, if X_t drops down into a domain $(0, l)$ because of a large scale demand for the system, the phase instantaneously changes to the inflow one with rate $\alpha_1(x)$. In two cases stated above the system can be switched without any loss of time. When the demand larger than the present level occurs, the storage becomes empty and the system is ruined. If a large scale demand happens in the inflow phase, the inflow phase is continued without the case of large scale demand dropping down below the zero level. Once the ruin occurs, $X(t)$ remains at level zero in a certain period according to an exponential distribution with parameter ν_0 . Immediately after that period, the inflow phase begins.

For this process, in the next section, we define the ruin probability and we constitute the integro-differential equations.

1.1 Integro-partial differential equations

For the model above, the states of the storage process are classified into four categories:

$$
\begin{cases}
(\xi(t), X(t)) = (1, x) & \text{if the process is in inflow phase and the storage level is } x \text{ at time } t,\ (0 < x < L)\ , \\
(\xi(t), X(t)) = (0, x) & \text{if the process is in outflow phase and the storage level is } x \text{ at time } t,\ (l < x < L)\ . \\
(\xi(t), X(t)) = 0 & \text{if the storage process is ruined at time } t, \\
(\xi(t), X(t)) = L & \text{if the storage is full at time } t,
\end{cases}
$$

where $\xi(t)$ indicates the present phase.

Thus we constitute the Markov process $\{(\xi(t), X(t)) : t \ge 0\}$.

Now, we define its probability distribution for $i = 0, 1$ and $t \ge 0$:

$$p_t(i,x) = P[(\xi(t), X(t)) = (i,x)] \ ,$$
$$P_t(0) = P[(\xi(t), X(t)) = 0] \ ,$$
$$P_t(L) = P[(\xi(t), X(t)) = L] \ .$$

Hence we have the following integro-partial differential equations with respect to $p_t(i,x)$.

For $p_t(1,x)$ $(0 < x < L)$

$$\frac{\partial p_t(1,x)}{\partial t} + \alpha_1(t)\frac{\partial p_t(1,x)}{\partial x} = -\lambda\{p_t(1,x) - \int_0^{L-x} p_t(1,x+y)dF(y)\}$$
$$+\lambda I_{(0,l)}(x)\{P_t(L)f(L-x) + \int_0^{L-x} p_t(0,x+y)dF(y)\} \ , \quad (1)$$

and for $p_t(0,x)$ $(l < x < L)$

$$\frac{\partial p_t(0,x)}{\partial t} - \alpha_0(x)\frac{\partial p_t(0,x)}{\partial x} = -\lambda\{p_t(0,x) - P_t(L)f(L-x) - \int_0^{L-x} p_t(0,x+y) \ \mathrm{dF(y)}\} \quad (2)$$

where $I_{0,l}(x) = 1$ if $x \in (0,l)$; $=0$ otherwise.

For the process with steady state we impose the condition on the process in the next subsection.

1.2 Integro-differential equation

Assuming that $\lim_{t\to\infty} p_t(i,x) = p(i,x)$ $(i = 0,1)$, $\lim_{t\to\infty} P_t(0) = P_0$ and $\lim_{t\to\infty} P_t(L) = P_L$ exist, we consider the steady state of this process.

Hence we have the following integro-differential equations from (1) and (2).

For $p(1,x)$ $(0 < x < L)$

$$\alpha_1(t)\frac{dp(1,x)}{dx} = -\lambda\{p(1,x) - \int_0^{L-x} p(1,x+y)dF(y)\}$$
$$+\lambda\{I_{(0,l)}(x)\{P_L \cdot f(L-x) + \int_0^{L-x} p(0,x+y)dF(y)\}\} \ , \quad (3)$$

and for $p(0,x)$ $(l < x < L)$

$$-\alpha_0(x)\frac{dp(0,x)}{dx} = -\lambda\{p(0,x) - P_L \cdot f(L-x) - \int_0^{L-x} p(0,x+y)dF(y)\} \quad (4)$$

To solve these equations, we need the boundary conditions:

$$\nu_0 P_0 = \lambda\int_0^{L}\{p(0,y) + p(1,y)\}\{1-F(y)\}dy + \lambda P_L\{1-F(L)\}, \quad (5)$$
$$(\nu_L + \lambda)P_L = \alpha_1(L-)p(1,L-), \quad (6)$$
$$\nu_L P_L = \alpha_0(L-)p(0,L-), \quad (7)$$
$$p(0,l+) + p(1,l-) = \int_0^{L-l}\{p(0,l+u) + p(1,l+u)\}f(u)du + P_L \cdot f(L-l). \quad (8)$$

Note that we suppose $p(0,x)=0$ for $0<x<l$.

1.3 Analytical Solutions for the storage process}

In this sub-section we present the analytical solutions for (3) and (4). First we have the following Theorem concerning with $p(0,x)$ in the outflow phase.

Theorem 1 *If we take* $\alpha_0(x)=\alpha_0$ *for* $l<x<L$ *then* $p(0,x)$ *is obtained as follows:*

$$p(0,x)=B(x)+\int_0^{L-x} e^{\delta_0 y}B(x+y)dM_0(y) \tag{9}$$

where

$$B(x)=A_0(x)-\frac{\lambda}{\alpha_0}\int_0^{L-x} A_0(x+y)\{1-F(y)\}dy, \tag{10}$$

$$A_0(x)=p(0,L-)+\frac{\lambda}{\alpha_0}P_L\cdot F(L-x), \tag{11}$$

$$M_0(y)=\sum_{n=1}^{\infty}H_0^{2n*}(y),H_0(y)=\frac{\lambda}{\alpha_0}\int_0^y e^{-\delta_0 y}\{1-F(y)\}dy \tag{12}$$

and δ_0 *is the unique solution of the equation*

$$\int_0^{\infty}\frac{\lambda}{\alpha_0}e^{-\delta_0 x}\{1-F(x)\}dx=1. \tag{13}$$

($H_0^{2n*}(y)$ *is the* $2n$-th *fold convolution of* $H_0(y)$.)

Next we have the following Theorem concerning with $p(1,x)\,(l<x<L)$ in the inflow phase.

Theorem 2 *If we take* $\alpha_1(x)=\alpha_1$ *for* $l<x<L$ *then* $p(1,x)$ *is obtained as follows:*

$$p(1,x)=p(1,L-)\{1+\int_0^{L-x} e^{\delta_1 y}dM_1(y)\}\ , \tag{14}$$

where

$$M_1(y)=\sum_{n=1}^{\infty}H_1^{n*}(y),H_1(y)=\frac{\lambda}{\alpha_1}\int_0^y e^{-\delta_1 y}\{1-F(y)\}dy \tag{15}$$

and δ_1 *is the unique solution of the equation*

$$\int_0^{\infty}\frac{\lambda}{\alpha_1}e^{-\delta_1 x}\{1-F(x)\}dx=1. \tag{16}$$

Next we have $p(1,x)$ for $0<x<l$.

Theorem 3 *If we take* $\alpha_1(x)=\alpha_2$ *for* $0<x<l$ *then* $p(1,x)$ *is obtained as follows:*

$$p(1,x)=A_1(x)+\int_0^{L-x} e^{\delta_2 y}A_1(x+y)dM_2(y), \tag{17}$$

where

$$A_1(x)=p(1,l-)-\frac{\lambda}{\alpha_2}P_L\{F(L-l)-F(L-x)\}$$

$$-\frac{\lambda}{\alpha_2}\int_x^l\int_l^L \{p(0,u)+p(1,u)\}f(u-v)dudv \ , \tag{18}$$

$$M_2(y)=\sum_{n=1}^{\infty}H_2^{n*}(y), H_2(y)=\frac{\lambda}{\alpha_2}\int_0^y e^{-\delta_2 y}\{1-F(y)\}dy \tag{19}$$

and δ_2 *is the unique solution of the equation*

$$\int_0^{\infty}\frac{\lambda}{\alpha_2}e^{-\delta_2 x}\{1-F(x)\}dx=1. \tag{20}$$

To get the ruin probability, we need to obtain the probability P_L by the following relation.

$$\int_0^L \{p(0,x)+p(1,x)\}dx+P_L+P_0=1, \tag{21}$$

where P_0 is expressed by P_L using the boundary condition.

Finally we have the ruin probability by (5). In order to express $p(1,l-)$ in (18) by P_L, we use (8) and (9). Thus we have

$$\begin{aligned} p(1,l-)=&\int_0^{L-l}\{p(0,l+u)+p(1,l+u)\}dF(u)+P_L\cdot f(L-l) \\ &-\Bigg[A_0(l+)-\frac{\lambda}{\alpha_0}\int_0^{L-l}A_0(l+y)\{1-F(y)\}dy+\int_0^{L-l}e^{\delta_0 y}A_0(l+y)dM_0(y) \\ &-\frac{\lambda}{\alpha_0}\int_0^{L-l}e^{\delta_0 y}\int_0^{L-(l+y)}A_0(l+u)\{1-F(u)\}dM_0(y)\Bigg]. \end{aligned} \tag{22}$$

Since the all probabilities are expressed by P_L, we have the following theorem for the ruin probability.

Theorem 4 *If we set*

$$p^*(0,x)=\frac{p(0,x)}{P_L},\ p^*(1,x)=\frac{p(1,x)}{P_L}\ \textit{and}\ P_0^*=\frac{P_0}{P_L},$$

then the ruin probability is obtained as

$$P_0=\frac{\lambda P_L}{v_0}\left[\int_0^L\{I_{(l,L)}(y)p^*(0,y)+p^*(1,y)\}\{1-F(y)\}dy+1-F(L)\right], \tag{23}$$

where

$$P_L=\left[\int_0^L\{p^*(0,x)+p^*(1,x)\}dx+P_0^*+1\right]^{-1}. \tag{24}$$

1.4 The case of exponential demands

We suppose the large scale demand occurs according to the Poisson distribution with parameter λ and its amount has the Exponential distribution with parameter μ.

In this case the quantities $M_i(x)\,(i=1,2)$ and $M_0(x)$ are given by

$$dM_i(x)=\frac{\lambda}{\alpha_i}dx\,(i=1,2), \tag{25}$$

$$dM_0(x)=\frac{\lambda}{2\alpha_0}e^{-\frac{\lambda}{\alpha_0}x}\left(e^{\frac{\lambda}{\alpha_0}x}-e^{-\frac{\lambda}{\alpha_0}x}\right)dx. \tag{26}$$

Furthermore $\delta_i\ (i=0,1,2)$ are obtained from (13), (16) and (20), respectively.

$$\delta_i=\frac{\lambda}{\alpha_i}-\mu\ (i=0,1,2).$$

By Theorem 1 and (7), we have $p(0,x)$ for the outflow phase $(l<x<L)$ as follows:

$$p(0,x)=\frac{\nu_L}{\alpha_0}P_L+\frac{\lambda}{\alpha_0}\left(1+\frac{\lambda}{\alpha_0\mu}\right)F(L-x)P_L+\left\{\frac{\nu_L}{\alpha_0}P_L-e^{-\mu(L-x)}\right\}(L-x)+Q(x), \tag{27}$$

where

$$\begin{aligned}Q(x)=&\left[\frac{\nu_L}{\alpha_0}\left\{\frac{\lambda}{2\alpha_0\delta_0}\left(1-\frac{1}{\delta_0}\right)\left(e^{\delta_0(L-x)}-1\right)\right.\right.\\&+\frac{\lambda}{2(\lambda+\alpha_0\mu)}\left(1-\frac{\alpha_0}{\lambda+\alpha_0\mu}\right)\left(e^{-(\frac{\lambda}{\alpha_0}+\mu)(L-x)}-1\right)+\left.\frac{\lambda}{2\alpha_0}\left(\frac{1}{\delta_0}+\frac{\alpha_0}{\lambda+\alpha_0\mu}\right)(L-x)\right\}\\&+\frac{\lambda}{2\alpha_0^2\delta_0}\left(\lambda-\frac{1}{\alpha_0\mu}\right)\left(e^{\delta_0(L-x)}-1\right)\\&+\frac{\lambda}{2\alpha_0(\lambda+\alpha_0\mu)}\left(\lambda-\frac{1}{\alpha_0\mu}\right)\left(e^{-(\frac{\lambda}{\alpha_0}+\mu)(L-x)}-1\right)-\frac{\lambda}{2\alpha_0}\left(e^{\delta_0(L-x)}-e^{-\mu(L-x)}\right)\\&\left.-\frac{\lambda}{2\alpha_0}\left(e^{-(\frac{\lambda}{\alpha_0}+\mu)(L-x)}-e^{-\mu(L-x)}\right)+\frac{1}{2\alpha_0^2}\left(e^{\delta_0(L-x)}+e^{-(\frac{\lambda}{\alpha_0}+\mu)(L-x)}-2\right)\right]P_L\\&-(L-x)e^{-\mu(L-x)}-\frac{\alpha_0}{2\lambda}\left(e^{\delta_0(L-x)}-e^{-\frac{\lambda}{\alpha_0}(L-x)}\right).\end{aligned} \tag{28}$$

Next, by Theorem 2 and (6), we have $p(1,x)$ for the inflow phase $(l<x<L)$.

$$p(1,x)=\frac{\nu_L+\lambda}{\alpha_1}\left\{1-\frac{\lambda}{\alpha_1\delta_1}+\frac{\lambda}{\alpha_1\delta_1}e^{\delta_1(L-x)}\right\}P_L \tag{29}$$

For $p(1,x)\,(0<x<l)$, $A_1(x)$ in (18) is obtained as follows:

$$\begin{aligned}A_1(x)=&\,p(1,l-)-\frac{\lambda}{\alpha_2}\left(e^{-\mu(L-x)}-e^{-\mu(L-l)}\right)P_L\\&-\frac{\lambda}{\alpha_2}(e^{\mu l}-e^{\mu x})\left[\frac{\nu_L+\lambda}{\alpha_1}\left\{\frac{1}{\mu}\left(1-\frac{\lambda}{\alpha_1\delta_1}\right)\left(e^{-\mu l}-e^{-\mu L}\right)+\frac{e^{\delta_1 L}}{\delta_1}\left(e^{-\frac{\lambda}{\alpha_1}l}-e^{-\frac{\lambda}{\alpha_1}L}\right)\right\}P_L\right.\end{aligned}$$

$$+\frac{\nu_L}{\alpha_0}\left\{\frac{1}{\mu}e^{-\mu l}(L-l)+\frac{1}{\mu}\left(1-\frac{1}{\mu}\right)\left(e^{-\mu l}-e^{-\mu L}\right)\right\}P_L$$

$$+\frac{\lambda}{\alpha_0}\left(1+\frac{\lambda}{\alpha_0\mu}\right)\left\{\frac{1}{\mu}\left(e^{-\mu l}-e^{-\mu L}\right)-e^{-\mu L}(L-l)\right\}P_L$$

$$\left.-\frac{1}{2}e^{-\mu L}(L-l)^2+\int_l^L e^{-\mu u}Q(u)du\right],$$

where

$$\int_l^L e^{-\mu u}Q(u)du=\left[\frac{\lambda}{2\alpha_0\delta_0}\left\{\frac{\alpha_0}{\lambda}e^{\delta_0 L}\left(e^{-\frac{\lambda}{\alpha_0}l}-e^{-\frac{\lambda}{\alpha_0}L}\right)-\frac{1}{\mu}\left(e^{-\mu l}-e^{-\mu L}\right)\right\}\right.$$

$$\cdot\left\{\frac{\nu_L}{\alpha_0}\left(1-\frac{1}{\delta_0}\right)+\frac{1}{\alpha_0}\left(\lambda-\frac{1}{\alpha_0\mu}\right)\right\}$$

$$+\frac{\lambda}{2(\lambda+\alpha_0\mu)}\left\{\frac{\alpha_0}{\lambda}e^{-(\frac{\lambda}{\alpha_0}+\mu)L}\left(e^{\frac{\lambda}{\alpha_0}L}-e^{\frac{\lambda}{\alpha_0}l}\right)-\frac{1}{\mu}\left(e^{-\mu l}-e^{-\mu L}\right)\right\}$$

$$\cdot\left\{\frac{\nu_L}{\alpha_0}\left(1-\frac{\alpha_0}{\lambda+\alpha_0\mu}\right)+\frac{1}{\alpha_0}\left(\lambda-\frac{1}{\alpha_0\mu}\right)\right\}$$

$$+\frac{\lambda\nu_L}{2\alpha_0^2}\left(\frac{1}{\delta_0}+\frac{\alpha_0}{\lambda+\alpha_0\mu}\right)\left\{(L-l)e^{-\mu l}-\frac{1}{\mu^2}\left(e^{-\mu l}-e^{-\mu L}\right)\right\}$$

$$+\frac{e^{-\mu L}}{2}\left(1+e^{-\frac{\lambda}{\alpha_0}(L-l)}\right)-e^{\delta_0 L-\frac{\lambda}{\alpha_0}l}+\frac{\lambda}{\alpha_0}e^{-\mu L}(L-l)$$

$$\left.+\frac{1}{2\alpha_0}\left\{\frac{\alpha_0}{\lambda}e^{-\frac{\lambda}{\alpha_0}l}\left(e^{\delta_0 L}-e^{-(\frac{\lambda}{\alpha_0}+\mu)L}\right)-2\right\}\right]P_L$$

$$-\frac{e^{-\mu L}}{2}(L-l)^2-\frac{\alpha_0^2}{2\lambda^2}e^{-\mu L}\left(e^{\frac{\lambda}{\alpha_0}L}+e^{-\frac{\lambda}{\alpha_0}(L+l)}-2\right). \tag{30}$$

For the second term of the right hand side of (17) we have the following.

$$\int_0^{l-x}e^{\delta_2 y}A_1(x+y)dy=\frac{\lambda}{\alpha_2\delta_2}(e^{\delta_2(l-x)}-1)\left[p(1,l-)+\int_l^L e^{-\mu u}Q(u)du\right.$$

$$\left.-\frac{\lambda}{\alpha_2}e^{\mu l}\left(\frac{\nu_L}{\alpha_0}\beta_0+\frac{\nu_L+\lambda}{\alpha_1}\beta_1+\beta_L\right)P_L\right]$$

$$+\frac{\lambda^2}{\alpha_2^2(\delta_2+\mu)}(e^{(\delta_2+\mu)(l-x)}-1)\left[e^{\mu x}\left(\frac{\nu_L}{\alpha_0}\beta_0+\frac{\nu_L+\lambda}{\alpha_1}\beta_1\right)\right.$$

$$\left.+\left\{\beta_L-\left(e^{-\mu(L-x)}-e^{-\mu(L-l)}\right)\right\}P_L\right], \tag{31}$$

where

$$\beta_0 = \frac{1}{\mu}e^{-\mu l}(L-l) + \frac{1}{\mu}\left(1-\frac{1}{\mu}\right)(e^{-\mu l} - e^{-\mu L}),$$

$$\beta_1 = \frac{1}{\mu}\left(1-\frac{\lambda}{\alpha_1\delta_1}\right)(e^{-\mu l} - e^{-\mu L}) + \frac{e^{\delta_1 L}}{\delta_1}\left(e^{-\frac{\lambda}{\alpha_1}l} - e^{-\frac{\lambda}{\alpha_1}L}\right),$$

$$\beta_L = \frac{\lambda}{\alpha_0}\left(1+\frac{\lambda}{\alpha_0\mu}\right)\left\{\frac{1}{\mu}(e^{-\mu l} - e^{-\mu L}) - e^{-\mu L}(L-l)\right\} - \frac{1}{2}e^{-\mu L}(L-l)^2.$$

2　Renewal process

In the same process above we evaluate the right hand side of (4). First we note the following relation.

$$-p(0,x) + \int_0^{L-x} p(0,x+y)f(y)dy = \frac{d}{dx}\int_0^{L-x} p(0,x+y)\{1-F(y)\}dy \tag{32}$$

Using this relation we rewrite (4) as:

$$p(0,x) = A_0(x) - \frac{\lambda}{\alpha_0}\int_0^{L-x} p(0,x+y)\{1-F(y)\}dy\ . \tag{33}$$

Since it is not the proper renewal function, we need the Tijms' method.

Using δ_0 defined by (13), we define the distribution function:

$$H_0(x) = \begin{cases} \dfrac{\lambda}{\alpha_0}\displaystyle\int_0^x e^{-\delta_0 x}\{1-F(y)\}dy\ (x>0), \\ 0\ (x \le 0). \end{cases} \tag{34}$$

2.1　Stabdard renewal equation

Now we have the following standard renewal equation in Theorem1, Theorem 2 and Theorem 3.

$$\begin{aligned} M_i(y) &= H_i^{1*}(y) + \sum_{n=1}^{\infty} H_i^{(n+1)*}(y) \\ &= H_i(y) + \int_0^{\infty} M_i(y-x)dH_i(x)\ (i=1,2), \end{aligned}$$

and

$$\begin{aligned} M_0(y) &= H_0^{2*}(y) + \sum_{n=1}^{\infty} H_0^{(2n+2)*}(y) \\ &= H_0^{2*}(y) + \int_0^{\infty} M_0(y-x)dH_0(x). \end{aligned}$$

It goes without saying that it is desirable to evaluate the convolution directly. If $F(x)$ is assumed to be the exponential distribution, we can evaluate the convolution easily.

First, using $\delta_i = \dfrac{\lambda}{\alpha_i} - \mu \ (i=1,2)$, we have

$$H_i(x) = 1 - e^{-\frac{\lambda}{\alpha_i}x}.$$

And it is easily to get $M_i(x)\,(i=1,2)$ as follows:

$$\begin{aligned} M_i(x) &= \sum_{k=1}^{\infty} H_i^{k*}(x) \\ &= \int_0^x \sum_{k=1}^{\infty} H_i^{(k-1)*}(x-u) \frac{\lambda}{\alpha_i} e^{-\frac{\lambda}{\alpha_i}u} du \\ &= \frac{\lambda}{\alpha_i} \int_0^x \{M_i(x-u)+1\} e^{-\frac{\lambda}{\alpha_i}u} du. \end{aligned}$$

Let us denote by $m_i(s)$ Laplace Transform of $M_i(x)$. Then we have

$$m_i(s) = \frac{\lambda}{\alpha_i} \cdot \frac{1}{s^2}.$$

By use of Inverse Transform, we have the following Proposition.

Proposition 1

$$M_i(x) = \frac{\lambda}{\alpha_i} x \ (i=1,2).$$

For $M_0(x)$, we need to evaluate G_0^{2*} as follows:

$$G_0^{2*}(x) = \frac{\lambda^2}{\alpha_0^2 \mu} \left(\frac{1}{\mu} - \frac{1}{\mu} e^{-\mu x} - x e^{-\mu x} \right).$$

Using $\delta_0 = \dfrac{\lambda}{\alpha_0} - \mu$, we get the following.

$$dH_0^{2k*}(x) = \frac{\lambda^2}{\alpha_0^2} x e^{-(\delta_0 + \mu)x} dx,$$

$$M_0(x) = \frac{\lambda^2}{\alpha_0^2} \int_0^x \{M_0(x-u)+1\} u e^{-\frac{\lambda}{\alpha_0}u} du.$$

In a similar fashion described above, we have the Proposition 2.

Proposition 2

$$M_0(x)=\frac{1}{4}e^{-\frac{2\lambda}{\alpha_0}x}+\frac{\lambda}{2\alpha_0}x-\frac{1}{4}.$$

3　Recursive equation

In this section, we consider the risk reserve process with exponential type claims and we find the non-ruin probability depending on the initial state in finite time. Let us denote by $U(t)$ the reserve level at time t, where $\{U(t)\}_{t\geq 0}$ is called the risk reserve process. If the reserve level is zero, the process ruins. The fluctuation of $U(t)$ is controlled by three elements : the claim inter-arrival time, the claim size and the premium rate. Let us assume that the claim inter-arrival time and the claim size are independent and identically distributed random variables, respectively. We also assume the premium rate is a constant. In the same way of Mikosch [7] we introduce the notation :

$$\begin{cases} U(t) & : \text{ the reserve level at time } t, \\ u & : \text{ the initial reserve level } (u=U(0)>0), \\ X_n & : \text{ the } n\text{-th claim size}, \\ T_n & : \text{ the } n\text{-th claim arrival time } (T_0=0), \\ W_n & : \text{ the claim inter-arrival time} \\ & \qquad \text{between } (n-1)\text{-th and n-th claim arrival times}, \\ c & : \text{ the premium rate}, \end{cases}$$

where X_n and W_n are independent.

The total claim amount process $\{S(t)\}_{t\geq 0}$ and the premium income $I(t)$ are defined as follows :

The total claim amount process $\{S(t)\}_{t\geq 0}$ is define as

$$S(t)=\sum_{n=1}^{N(t)}X_n \quad (t\geq 0),$$

where $\{N(t)\}_{t\geq 0}$ is the claim number process defined by

$$N(t)=\max\{n\geq 1: T_n\leq t\} \quad (t\geq 0).$$

We define the premium income by I(t) = ct, which is the accumulated income by time t.
Therefore, we obtain the expression of risk reserve process $\{U(t)\}_{t\geq 0}$ as follows :

$$U(t)=u+I(t)-S(t), \quad (t\geq 0).$$

3.1　Mathematical model for the ruin probability

We make a mathematical model to get the non-ruin probability in finite time (Doi[6]).

In the risk reserve process $\{U(t)\}_{t\geq 0}$, the ruin can occur only at the time $t=T_n$ for some $n\geq 1$, since $\{U(t)\}_{t\geq 0}$ linearly increases in the intervals $[T_n, T_{n+1})$. We call the sequence $\{U(T_n)\}_{n\geq 0}$ the skeleton process of the risk reserve process $\{U(t)\}_{t\geq 0}$ (see Mikosch [7]). By use of the skeleton process, we can express the event $\{ruin\}$ in terms of the inter-arrival times W_n, the claim sizes X_n, the initial reserve level and the premium rate c, as follows :

$$\{\text{ruin}\} = \left\{\inf_{t>0} U(t) < 0\right\}$$

$$= \left\{\inf_{n\geq 1}[u + I(T_n) - S(T_n)] < 0\right\}$$

$$= \left\{\inf_{n\geq 1}\left[u - \sum_{i=1}^{n}(X_i - cW_i)\right] < 0\right\}.$$

Now, we define

$$Z_n = X_n - cW_n, \quad (n\geq 1),$$
$$S_n = Z_1 + \cdots + Z_n, \quad (n\geq 1, S_0 = 0).$$

In this section, we propose R|Ex|Ex and R|Ex|Er models where R means the risk reserve process, the first Ex means that the claim inter-arrival time W_n has an exponential distribution with rate λ and the next Ex means that the claim size X_n has an exponential distribution with rate μ and Er:Erlang distribution with parameter μ and phase k. In what follows, we omit the subscript n.

3.2 pdf. for the process

We find the probability distribution of random variable Z for this model. First, let $Y=cW$, which has the probability density function as follows :

$$f_Y(y) = \begin{cases} \dfrac{\lambda}{c} e^{-\frac{\lambda}{c}y} & (y\geq 0) \\ 0 & (y<0). \end{cases}$$

Next, let us denote

$$\begin{cases} Z = X - Y \\ V = Y. \end{cases}$$

Since X and Y are independent random variables, we obtain the joint probability density function with respect to Z and V

$$f_{ZV}(z,v) = \mu e^{-\mu(z+v)} \cdot \frac{\lambda}{c} e^{-\frac{\lambda}{c}v},$$

where the domain of v is

$$\begin{cases} 0 \leq v < \infty & (z \geq 0) \\ -z \leq v < \infty & (z < 0). \end{cases}$$

We obtain the probability density function $g(z)$ of Z as follows :

$$g(z) = \begin{cases} \dfrac{\lambda\mu}{\lambda + c\mu} e^{-\mu z} & (z \geq 0) \\ \dfrac{\lambda\mu}{\lambda + c\mu} e^{\frac{\lambda}{c} z} & (z < 0). \end{cases} \tag{35}$$

3.3 Recursive equation for the non-ruin or ruin probability

We denote by $r_n(u, c)$ the non-ruin probability that the risk reserve process does not ruin till n-th claim arrival time given the initial reserve level u and the premium rate c, that is,

$$rn(u,c) \tag{36}$$

$$= P(Z_1 < u, Z_2 < u - S_1, \cdots, Z_n < u - S_{n-1} \mid U(0) = u, T_1 < T_2 < \cdots < T_n < \infty). \tag{37}$$

Now we obtain the two Theorems for R|Ex|Ex model and R|Ex|Er model.

Theorem 5 *For* R|Ex|Ex *model, we obtain the probability* $r_n(u, c)$ *as follows* :

$$r_0(u,c) = 1,$$

$$r_n(u,c) = r_{n-1}(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda + c\mu}\right)^n e^{-u\mu} \sum_{i=0}^{n-1} K_{n,n-i} \frac{u^i}{i!}\left(\frac{c}{\lambda + c\mu}\right)^{n-i-1}, \quad (n \geq 1) \tag{38}$$

where

$$\begin{cases} K_{n,1} = 1, & (n \geq 1) \\ K_{n,n} = K_{n,n-1}, & (n \geq 2) \\ K_{n,l} = K_{n-1,l} + K_{n,l-1}, & (n \geq 3, 2 \leq l \leq n-1) \\ K_{n,l} = 0, & \text{(others)}. \end{cases} \tag{39}$$

Theorem 6 *For* R|Ex|Er *model, we obtain the non-ruin probability in finite time* $r_n^{(k)}(u, c)$ *as follows*:

$$r_0^{(k)}(u,c) = 1,$$

$$r_n^{(k)}(u,c) = r_{n-1}^{(k)}(u,c) - \left(\frac{\lambda}{c}\right)^n \alpha^{-n} A^{k(n-1)} e^{-u\beta} \sum_{i=0}^{kn-1} K_{n,kn-i}^{(k)} \frac{(u\alpha)^i}{i!}, \quad (n \geq 1) \tag{40}$$

where

$$\begin{cases} \alpha = k\mu + \dfrac{\lambda}{c} \\ \beta = k\mu \\ A = \alpha^{-1}\beta, \end{cases} \tag{41}$$

and

$$\begin{cases} K_{n,1}^{(k)} = A^{k-1}, & (n \geq 1) \\ K_{1,m}^{(k)} = \sum_{j=1}^{m} A^{k-j}, & (2 \leq m \leq k) \\ K_{n,kn-j}^{(k)} = K_{n,k(n-1)}^{(k)}, & (n \geq 2, 1 \leq j \leq k) \\ K_{n,m}^{(k)} = K_{n-1,m}^{(k)} + K_{n,m-1}^{(k)}, & (n \geq 2, 2 \leq m \leq k(n-1)) \\ K_{n,m}^{(k)} = 0, & \text{(others).} \end{cases} \tag{42}$$

The ruin probability for R|Ex|Ex model is given as follows:

Theorem 7

$$p_n(u,c) = \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda + c\mu}\right)^n e^{-u\mu}\sum_{i=0}^{n-1} K_{n,n-i}\frac{u^i}{i!}\left(\frac{c}{\lambda + c\mu}\right)^{n-i-1} \quad (n \geq 1). \tag{43}$$

Also the ruin probability for R|Ex|Er (phase k) model:

Theorem 8

$$p_n^{(k)}(u,c) = \left(\frac{\lambda}{c}\right)^n \alpha^{-n} A^{k(n-1)} e^{-u\beta}\sum_{i=0}^{kn-1} K_{n,kn-i}^{(k)}\frac{(u\alpha)^i}{i!} \quad (n \geq 1). \tag{44}$$

4 Arrival acceptance window

In this section, we consider a queue where there is an arrival acceptance window generated through assigning arrival times to all customers. If a customer needs to receive the service, he has to arrive during his assigned window. A customer that cannot enter the system within the arrival acceptance window is not offered the service and he must leave the system. The arrival acceptance window is generated through assigning arrival times to customers arbitrarily. At most one customer can be accepted to the queue in each window. Although the customer's arrival time is scheduled, actual one may be delayed. So we assume that the delay-arrival time is arbitrarily distributed. And the service time of accepted customer is assumed to be exponentially distributed. We refer this queueing system with one server as $GI \mid GI/M/1$. Queueing models with the arrival acceptance window of this kind can be found in various situations (Doi, Chen and Ōsawa [8], Ōsawa, Doi, Chen and Shima [9]). Such a typical example is found in production systems of many factories and in scheduled arrival systems of medical clinics.

4.1 Mathematical model

We define a mathematical model for the $GI \mid GI/M/1$. The n, th window is generated at time t_n and the n, th customer is scheduled to enter the system during the time interval [$t_n\, t_{n+1}$), where $0 \le t_1 < t_2 < \cdots$. Let τ_n be the actual arrival time of the n th customer where $\tau_n \ge t_n$. The customer that does not enter the system by t_{n+1} , i.e., $\tau_n > t_{n+1}$, is refused to be offered the service. That is, the customer must leave the system and never come back again. We assume that durations of windows $t_{n+1} - t_n$ are independent and identically distributed random variables with a distribution function $A(t)$ which has the finite mean $1/\lambda$. We also assume delay-arrival times $\tau_n - t_n$ are $i.i.d.$ with a distribution function $B(t)$ having the finite mean $1/\nu$. Note that the acceptance probability of the window is $\int_0^\infty B(t)\, dA(t)$. The service times of accepted customers have a common exponential distribution with the parameter μ . And the system has one server.

Let $\xi(t)$ be the number of customers in the system at time t . We choose t_n as the embedded points for the system size process $\{\xi(t);\, t \ge 0\}$ and denote $\xi_n = \xi(t_n - 0)$. The embedded process $\Xi = \{\xi_n; n = 1, 2, \ldots\}$ constitutes a Markov chain.

For the $GI \mid GI/M/1$, Ōsawa et al.[9] have analyzed the system size process Ξ and obtained the necessary and sufficient conditions in which the limiting distribution of the process Ξ exists. In order to have the optimal arrival acceptance window size, we introduce costs to the queueing system. The system gets the benefit and incurs the waiting cost if a customer is offered the service. If a customer is not offered the service, the system incurs a lost-opportunity cost.

4.2 System size at scheduled times

Let us denote the transition probabilities for the process Ξ as :

$$P_{ij} = \mathbf{P}[\xi(n+1) = j \,|\, \xi(n) = i], \quad i, j \ge 0.$$

For $j \ge 2$ we obtain these as follows:

$$P_{ij} = \int_0^\infty \frac{e^{-\mu y}(\mu y)^{i+1-j}}{(i+1-j)!} B(y)dA(y) + \int_0^\infty \frac{e^{-\mu y}(\mu y)^{i-j}}{(i-j)!} \bar{B}(y)dA(y), \quad i \ge j,$$

$$P_{j-1,j} = \int_0^\infty e^{-\mu y} B(y)dA(y),$$

where $\bar{B}(y) = 1 - B(y)$.

In the similar way, we have the transition probabilities for $j = 1$ and 0 as follows:

$$P_{i1} = \int_0^\infty \bar{B}(y) \frac{e^{-\mu y}(\mu y)^{i-1}}{(i-1)!} dA(y) + \int_0^\infty \int_0^y B(t) \frac{e^{-\mu t}(\mu t)^{i-1}}{(i-1)!} \mu dt e^{-\mu(y-t)} dA(y)$$
$$+ \int_0^\infty \int_0^y \int_0^\tau \frac{e^{-\mu t}(\mu t)^{i-1}}{(i-1)!} \mu dB(\tau) d\tau e^{-\mu(y-\tau)} dA(y), \quad i \geq 1,$$
$$P_{01} = \int_0^\infty \int_0^y dB(\tau) e^{-\mu(y-\tau)} dA(y),$$
$$P_{i0} = \int_0^\infty \int_0^y \frac{\mu e^{-\mu t}(\mu t)^{i-1}}{(i-1)!} dt \bar{B}(y) dA(y)$$
$$+ \int_0^\infty \int_0^y B(t) \frac{e^{-\mu t}(\mu t)^{i-1}}{(i-1)!} \mu dt (1 - e^{-\mu(y-t)}) dA(y)$$
$$+ \int_0^\infty \int_0^y \int_0^\tau \frac{e^{-\mu t}(\mu t)^{i-1}}{(i-1)!} \mu dt dB(\tau)(1 - e^{-\mu(y-\tau)}) dA(y), \quad i \geq 1,$$
$$P_{00} = \int_0^\infty \bar{B}(y) dA(y) + \int_0^\infty \int_0^y dB(\tau)(1 - e^{-\mu(y-\tau)}) dA(y).$$

For any other cases, i.e., $j \geq i+2$, we have

$$P_{ij} = 0.$$

We obtain the transition probability matrix for the chain Ξ of the $GI/M/1$ type:

$$P = \begin{pmatrix} P_{00} & P_{01} & 0 & \cdot & \cdot & & & \\ P_{10} & P_{11} & a_0 & 0 & \cdot & \cdot & & \\ P_{20} & P_{21} & a_1 & a_0 & 0 & \cdot & & \\ P_{30} & P_{31} & a_2 & a_1 & a_0 & 0 & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

where a_k is defined by

$$a_k = \int_0^\infty \frac{e^{-\mu y}(\mu y)^k}{k!} B(y) dA(y) + \int_0^\infty \frac{e^{-\mu y}(\mu y)^{k-1}}{(k-1)!} \bar{B}(y) dA(y), k \geq 1,$$
$$a_0 = \int_0^\infty e^{-\mu y} B(y) dA(y). \tag{45}$$

By use of the fundamental theory of the $GI/M/1$ type matrix, we can analyze the process Ξ . Let us denote the Laplace Stieltjes transform concerning the distribution $A(x)$ as $a[s]$:

$$a[s] = \int_0^\infty e^{-sx} dA(x), \quad s \geq 0.$$

We define the generating function for a sequence $\{a_k, k \geq 0\}$: $\Phi(z)$ by (45), then, for $0 \leq z \leq 1$, we have

$$\Phi(z) = \int_0^\infty e^{-\mu(1-z)y} \{B(y) + z\bar{B}(y)\} dA(y). \tag{46}$$

We have the following.

Lemma 1 *The equation* $z = \Phi(z)$ *has a unique solution such that* $0 < z < 1$ *if and only if*

$$\rho \int_0^\infty B(y) dA(y) < 1 \text{, where } \rho = \lambda / \mu. \tag{47}$$

Throughout the paper, we denote as ζ the solution described in the lemma. Let us introduce the distribution functions F and I on $[0, \infty)$ defined by

$$F(x) = 1 - e^{-\mu(1-\zeta)x},$$
$$I(x) = 1.$$

For convenience, we define a function

$$\beta[H] = \int_0^\infty \int_0^y d(B \cdot H)(t)\, e^{-\mu(y-t)} dA(y),$$

where H is a distribution function on $[0, \infty)$. Then we can have the limiting distribution of the system size as follows.

Theorem 9 *The process* Ξ *is positive recurrent if and only if* $\rho \int_0^\infty B(y)\, dA(y) < 1$.
Moreover, under this condition, the limiting distribution of the system size process is given by

$$\pi_k = \lim_{n \to \infty} \mathbf{P}[\xi_n = k] = \begin{cases} 1 - \sigma, & (k = 0), \\ \sigma(1-\zeta)\zeta^{k-1}, & (k \geq 1), \end{cases} \tag{48}$$

where ζ is the unique solution defined as above and

$$\sigma = \frac{\beta[I]}{\int_0^\infty F(y) dA(y) + \beta[I] - \beta[F]}.$$

From this theorem, we can obtain the following corollary.

Corollary 10 *The mean stationary system size L and its variance* V_L *are given by*

$$L = \frac{\sigma}{1-\zeta},$$
$$V_L = \frac{\sigma(1+\zeta-\sigma)}{(1-\zeta)^2}.$$

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References

[1] Doi, M. and Ōsawa, H., A storage process with non-constant inflow and outflow, *Mathematical Modelling in Science and Technology*, *The Fourth International Conference*, Pergamon Press, pp.305-310, 1983.
[2] Doi, M., The expected ruin time for a storage process, *Mathematica Japonica*, Vol.51, No.1, pp.67-74, 2000.
[3] Doi, M., The ruin time for a risk reserve process, *Proceedings of the Conference CompStat 2002*, pp.1-2, 2002.
[4] Doi, M., Nagai, T. and Ōsawa, H., The ruin probability for a risk reserve process, *Proceedings of the 4-th ARS Conference of the IASC*, pp.82-84, 2002.
[5] Doi, M., Mean ruin time for the risk reserve process, *Bulletin of Research Institute of Economic Science Nihon Univ.*, Vol.37, pp.29-37, 2007.
[6] Doi, M., The ruin probability and the ruin time for the risk reserve process, *Bulletin of Research Institute of Economic Science Nihon Univ.*, Vol.48, pp.127-139, 2015.
[7] Mikosch, T., *Non-life insurance mathematics*, Springer, Berlin, 2004.
[8] Doi, M., Chen, Y. and Ōsawa, H., A queueing model in which arrival times are scheduled, *Operations Research Letters*, Vol.21, pp.249-252, 1997.
[9] Ōsawa, H., Doi, M., Chen, Y., Shima, C., A queueing model with scheduled arrivals, *International Journal of Information and Management Sciences*, Vol.11, pp.49-58, 2000.
[10] Neuts, M.F., *Matrix Geometric Solutions in Stochastic Models*, John Hopkins University Press, Baltimore, Maryland, 1981.