Lethargic Approximation: Banach and Fréchet Spaces Forming Bernstein Pairs

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Abstract

In this paper, we examine the Bernstein's Lethargy Theorem in the context of Banach and Fréchet spaces and define Bernstein pair. We introduce conditions under which a pair of Banach spaces form a Bernstein pair. Some open questions relating Bernstein pairs for Fréchet spaces are also presented.

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1 Introduction

One of the notable theorems used in the constructive theory of functions is Bernstein Lethargy Theorem (BLT) [7] and this paper is motivated by this theorem. For $f \in C([0,1])$, the sequence of best approximation is defined as

$$\rho(f, P_n) = \inf\{||f-p|| : p \in P_n\} \text{ for all n} \geq 1, \tag{1}$$

where P_n denotes the space of all polynomials of degree $\leq n$. Clearly

$$\rho(f, P_1) \geq \rho(f, P_2) \geq \cdots$$

and $\{\rho(f, P_n)\}$ form a non-increasing sequence of numbers. S. N. Bernstein in [7] proved that if $\{d_n\}_{n\geq 1}$ is a non-increasing null sequence (i.e., $\lim\limits_{n\to\infty} d_n = 0$) of positive numbers, then there exists a function $f \in C[0,1]$ such that

$$\rho(f, P_n) = d_n \text{ for all } n \geq 1.$$

This remarkable result is called Bernstein's Lethargy Theorem (BLT) and is used in the constructive theory of functions [24], and it has been applied to the theory of quasi analytic functions in several complex variables [22]. Note that the density of polynomials in $C[0,1]$ (the Weierstrass

Approximation Theorem) implies that

$$\lim_{n\to\infty} \rho(f, P_n) = 0.$$

However, the Weierstrass Approximation Theorem gives no information about the speed of convergence for $\rho(f, P_n)$. Bernstein's Lethargy Theorem has been extended replacing $C[0, 1]$ by arbitrary Banach space X and replacing P_n to arbitrary closed subspaces of X [9]. Furthermore, Tyuriemskih [26] showed that the sequence of errors of best approximation from x to Y_n, $\{\rho(x, Y_n)\}$, may converge to zero at an arbitrarily slow rate. More precisely, for any expanding sequence $\{Y_n\}$ of subspaces of X and for any null sequence $\{d_n\}$ of positive numbers, he constructed an element $x \in X$ such that

$$\lim_{n\to\infty} \rho(x, Y_n) = 0, \text{ and } \rho(x, Y_n) \geq d_n \text{ for all } n \geq 1.$$

However, it is also possible that the errors of best approximation $\{\rho(x, Y_n)\}$ may converge to zero arbitrarily fast, for results of this type see [5].

We refer the reader to [11] for an application of Tyuriemskih's Theorem to convergence of sequence of bounded linear operators.

We also refer to [1, 3, 5, 16] for other versions of Bernstein's Lethargy Theorem and to [2, 4, 17, 21, 28] for Bernstein's Lethargy Theorem for Fréchet spaces.

Given an arbitrary Banach space X, a strictly increasing sequence $\{Y_n\}$ of subspaces of X and a non-increasing null sequence $\{d_n\}$ of non-negative numbers, one can ask the question whether there exists $x \in X$ such that $\rho(x, Y_n) = d_n$ for each n. For a long time no sequence $\{d_n\}$ of this type was known for which such an element x exists for *all* possible Banach spaces X. The only known spaces X in which the answer is always "yes" are the Hilbert spaces (see [27]). For a general (separable) Banach space X, a solution x is known to exist whenever all Y_n are finite-dimensional (see [25]). Moreover, it is known that if X has the above property, then it is reflexive (see [27]).

2 Banach Spaces Forming Bernstein Pairs

Before we define the concept of Bernstein Pair, we need the define the approximation number.

Definition 1 *Let X and Y be Banach spaces. For every operator $T \in \mathcal{L}(X, Y)$ the n-th approximation number of T is defined as*

$$a_n(T) = \inf\{||T - S|| : S \in \mathcal{L}(X, Y), \text{ rank}\, S < n\}.$$

Clearly,

$$||T|| = a_1(T) \geq a_2(T) \geq \cdots \geq 0 \quad \text{and}$$

$$a_n(T) = \rho(T, \mathcal{F}_n),$$

where $\mathcal{F}_n$ is the space of bounded linear operators from X to Y with rank at most n. Approximation numbers are the largest s-numbers [20]. For operators on Hilbert spaces there are some well known properties of $a_n(T)$. Namely, if X and Y are Hilbert spaces, then $(a_n(T))$ is the sequence of singular numbers of T and moreover if T is a compact operator, then $a_n(T) = \lambda_n(T^*T)^{1/2}$, where

$$\lambda_1(T^*T) \geq \lambda_2(T^*T) \geq \cdots \geq 0$$

are the eigenvalues of T^*T ordered as above. We refer the reader to [10, 19] for general information about approximation and other s-numbers. We say T is *approximable* if $\lim_{n\to\infty} a_n(T) = 0$. Any approximable operator is compact, but the converse is not true due to the existence of Banach spaces without approximation property.

Definition 2 *Two Banach spaces X and Y are said to form a Bernstein pair (BP) if for any positive monotonic null sequence (d_n) there is an operator $T \in \mathcal{L}(X, Y)$ and a constant M depending only on T and (d_n) such that*

$$d_n \leq a_n(T) \leq M d_n \quad \textit{for all } n.$$

In this case we say that $(a_n(T))$ is equivalent to (d_n) and write (X, Y) to denote Bernstein pair, and in case $M = 1$, (X, Y) is called an exact Bernstein pair.

Here by $a_n(T)$ we mean the n-th approximation number of T defined above.

Theorem 3 *For any Hilbert space H, (H, H) form an exact Bernstein pair.*

Proof. If both spaces X and Y are equal to a Hilbert space H and if $\{d_n\}$ is a sequence decreasing to 0, then there exists *a compact operator* T on H such that

$$a_n(T) = d_n$$

(see [20]). Thus (H, H) form an exact Bernstein pair. ■

All of these desirable properties of $a_n(T)$ for operators between Banach spaces brings us to the following question:

Question: Suppose $\{d_n\}_{n\geq 1}$ is a non-increasing null sequence (i.e., $d_n \searrow 0^+$) of positive numbers. Does there exist $T \in \mathcal{L}(X, Y)$ such that the sequence $(a_n(T))$ behaves like the sequence $\{d_n\}$?

In other words, suppose $d_n \searrow 0^+$,

a) Does there exist $T \in \mathcal{L}(X, Y)$ such that $a_n(T) \geq d_n$ for any n?

b) Does there exist $T \in \mathcal{L}(X, Y)$ such that $a_n(T) = d_n$ for any n?

c) Does there exist $T \in \mathcal{L}(X, Y)$ and a constant M such that

$$\frac{d_n}{M} \leq a_n(T) \leq M d_n$$

for any n?

First results of this kind appeared in [14] in the context of Banach spaces with basis. They proved the following theorem.

Theorem 4 (C. Hutton, J. Morrell and J. Retherford [14]) *Suppose* $\{d_n\}_{n\geq 1}$ *is a non-increasing null sequence of positive numbers and* X *is a Banach space with basis. Then there exists* $T \in B(X)$ *such that*

$$d_m \leq a_m(T) \leq K d_m,$$

where the constant K *depends on* X*.*

Later these results sharpened as follows:

Theorem 5 (A. G. Aksoy and G. Lewicki [3]) *Suppose* $\{d_n\}_{n\geq 1}$ *is a non-increasing null sequence of positive numbers. In each of the following cases there exists* $T \in B(X, Y)$ *such that*

$$a_m(T) = d_m \quad \textit{for each } m$$

a) X *and* Y *has 1-unconditional basis (for example* $X = Y = \ell_p$ *for* $1 \leq p < \infty$ *and* $X = Y = c_0$*).*

b) $X \in \{c_0, \ell_\infty\}$ *and* Y *has 1-symmetric basis.*

c) $Y = \ell_1$ *and* X *has 1-symmetric basis.*

In [3] among other things, it was proved that if we assume (X, Y) is a Bernstein pair with respect to $\{a_n\}$ and suppose that a Banach space W contains an isomorphic and a complementary copy of X, and a Banach space V contains an isomorphic copy of Y, then (W, V) is a Bernstein Pair with respect to $\{a_n\}$. This implies that there are some natural pairs of Banach spaces that form a Bernstein pair as shown in the following.

Theorem 6 *For* $1 < p < \infty$ *and* $1 \leq q < \infty$*, the couple* $(L_p[0,1], L_q[0,1])$ *form a Bernstein Pair.*

Proof. This follows from the fact that (ℓ_2, ℓ_2) is a Bernstein pair with respect to $\{a_n\}$ ([19]) and the fact that for every p, $1 \leq p < \infty$, $L_p[0,1]$ contains a subspace isomorphic to ℓ_2 and complemented in $L_p[0,1]$ for $p > 1$ (see [29], page 85). ■

Example 7

a) If $(\ell_\infty, \ell_\infty)$ is BP, then (X, Y) is BP provided that both X and Y contains isomorphic copy of ℓ_∞ (Phillips Theorem).

b) If (c_0, c_0) is a BP, then (X, Y) is BP provided that both X and Y each contain an isomorphic copy of c_0 and X is separable (Sobczyk's Theorem).

c) If (ℓ_1, ℓ_1) is a BP, then (X, Y) is BP provided that both X and Y each contain an isomorphic copy of ℓ_1 and X is a non reflexive subspace of $L_1[0,1]$ (Pelczynski-Kadeč Theorem).

For the statements of Phillips, Sobczyk and Pelczynski-Kadeč Theorems we refer the reader to [12].

The most recent result on the rate of decay of approximation numbers can be found in [18], where it is shown that for two infinite dimensional Banach spaces X and Y and a non-increasing null sequence $\{d_n\}_{n\geq 1}$, there exists an approximable $T : X \to Y$ such that

$$||T|| \leq 2d_1 \quad \text{and} \quad d_m/9 \leq a_m(T) \leq 3d_{[m/4]} \quad \text{for any } m.$$

It is also worth mentioning that in [15] operators with prescribed eigenvalue sequences are constructed.

3 Fréchet Spaces and Bernstein Pairs

In this section we define Fréchet spaces, mention recent results of Bernstein's Lethargy Theorem for Fréchet spaces and pose questions whether or not two Fréchet spaces X and Y form a Bernstein pair. Fréchet spaces are locally convex spaces that are complete with respect to a translation invariant metric and they are generalization of Banach spaces.

Definition 8 *$(X, \|\cdot\|_F)$ is called a Fréchet space, if it is a metric linear space which is complete with respect to its F-norm $||\cdot||_F$ giving the topology. As usual by a F-norm we mean that $||\cdot||_F$ satisfies the following conditions [23]:*

1. *$||x||_F = 0$ if and only if $x = 0$,*
2. *$||\alpha x||_F = ||x||_F$ for all real or complex α with $||\alpha|| = 1$,*
3. *$||x + y||_F \leq ||x||_F + ||y||_F$.*

Many Fréchet spaces X can also be constructed using a countable family of semi-norms $||x||_k$, where X is a complete space with respect to this family of semi-norms. For example a translation invariant complete metric inducing the topology on X can be defined as

$$d(x,y) := \sum_{k=0}^{\infty} 2^{-k} \frac{||x-y||_k}{1+||x-y||_k} \quad \text{for } \ x,y \in X.$$

Clearly, every Banach space is a Fréchet space, and the other well known example of a Fréchet space is the vector space $C^{\infty}[0,1]$ of all infinitely differentiable functions $f : [0,1] \to \mathbb{R}$, where the semi-norm is

$$||f||_k = \sup\{|f^k(x)| : \, x \in [0,1]\}.$$

For more information about Frechét spaces the reader is referred to [23].
Bernstein's Lethargy Theorem (BLT) is also studied for Fréchet spaces.
The most recent result in this direction can be found in [2]. More precisely, let X be an infinite-dimensional Fréchet space and let $\mathcal{V} = \{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$. Let d_n be a decreasing sequence of positive numbers tending to 0. Under an additional natural condition on $\sup\{\text{dist}(x, V_n)\}$, we proved that there exists $x \in X$ and $n_o \in \mathbb{N}$ such that

$$\frac{d_n}{3} \leq \rho(x, V_n) \leq 3d_n$$

for any $n \geq n_o$.
Using the above-mentioned theorems of BLT type it is natural to examine Bernstein pairs in the context of Fréchet spaces. In other words we ask what are the conditions under which two Fréchet spaces X and Y to form a Bernstein pair? To be able to obtain results like Bernstein's Lethargy Theorem, one useful condition is to look at the deviation of the subset V_n from V_{n+1} as defined in the following.

Notation 9 *Let $(X, \|\cdot\|_F)$ be a Fréchet space and assume that $\mathcal{V} = \{V_n\}$ is a nested sequence of linear subspaces of X satisfying $\overline{V_n} \subset V_{n+1}$. Then the deviation of V_n from V_{n+1} is defined as*

$$d_{n,\mathcal{V}} = \sup\{\rho(v, V_n) : v \in V_{n+1}\}. \tag{2}$$

Here in this paper we assume that

$$d_{\mathcal{V}} = \inf\{d_{n,\mathcal{V}} : n \in \mathbb{N}\} > 0. \tag{3}$$

Necessity of this assumption is illustrated in the following example.

Example 10 *Let $X = \{x = (x_n) : x_n \in \mathbb{R}$ for any $n \in \mathbb{N}\}$ be equipped with the F-norm $\|x\|_F = \sum_{j=1}^{\infty} \frac{|x_j|}{2^j(1+|x_j|)}$. Let*

$$V_n = \{x \in X : x_k = 0 \quad \textit{for } k \geq n+1\}.$$

It is easy to see that for any $x \in X$

$$\rho(x, V_n) = \sum_{j=n+1}^{\infty} \frac{|x_j|}{2^j(1+|x_j|)} \leq \frac{1}{2^n}.$$

Let $d_n = \dfrac{2}{n}$ *and observe that for any* $x \in X$, $\rho(x, V_n) \leq \dfrac{1}{2^n} < \dfrac{2}{n}$. *Also observe that* $d_{n,\mathcal{V}} = \dfrac{1}{2^{n+1}}$, *which implies that* $d_{\mathcal{V}} = 0$.

Note that in case of Banach space X, we have $d_{n,\mathcal{V}} = +\infty$ because

$$\rho(tx, V_n) = t\rho(x, V_n)$$

and the supremum taking over all $v \in V_{n+1}$ and V_n is strictly included in V_{n+1}.
The next example is to show that there is a natural way to think of Fréchet spaces when $d_{n,\nu} = 1$.

Example 11 *Let* $(X, \|\cdot\|)$ *be a Banach space. Define in* X *an F-norm* $\|\cdot\|_F$ *by* $\|x\|_F = \|x\|/(1+\|x\|)$. *Then* $d_{n,\mathcal{V}} = 1$ *for any* $n \in \mathbb{N}$ *independently of* $\mathcal{V}$. *Because the mapping*

$$t \mapsto \frac{t}{1+t}$$

is increasing for $t > -1$ *and*

$$\rho_F(tx, V_n) := \frac{\rho(tx, V_n)}{1+\rho(tx, V_n)} \to 1$$

as $t \to \infty$.

Theorem 12 ([2]) *Let* X *be a Fréchet space and assume that* $\mathcal{V} = \{V_n\}$ *is a nested sequence of linear subspaces of* X *satisfying* $\overline{V_n} \subseteq V_{n+1}$, *where the closure is taken with respect to* $\|\cdot\|_F$. *Let* $d_{n,\mathcal{V}}$ *be defined as above and* $\{e_n\}$ *be a decreasing sequence of positive numbers satisfying*

$$\sum_{j=n}^{\infty} 2^{j-n}(\delta_j + e_j) < \min\{d_{n,\mathcal{V}}, e_{n-1}\}$$

with a fixed sequence of positive numbers δ_j. *Then, there exists* $x \in X$ *such that* $\rho(x, V_v) = d_n$ *for any* $n \in \mathbb{N}$.

Let X, Y be Fréchet spaces. Then, *n-th approximation number* of $T : X \to Y$ is defined as

$$a_n(T) = \inf\{||T - S||_F : S \in \mathcal{L}(X, Y), \text{rank } S < n\}.$$

We again have

$$||T||_F = a_1(T) \geq a_2(T) \geq \cdots \geq 0.$$

In the light of the Theorem given in [14], we know that if X is a Banach space with basis, then (X, X) forms a Bernstein pair, thus it is relevant to identify those Fréchet spaces which have basis.

Theorem 13 *Let X be a nuclear Fréchet space, then (X, X) forms a Bernstein Pair.*

Proof. It is known that nuclear Fréchet spaces have basis (see [13]) ■

Questions:

a) Let X and Y be Fréchet spaces, is it true that if we know that (X, Y) is an exact Bernstein pair with respect to $\{a_n\}$ and if we suppose that a Fréchet space W contains an isomorphic and complementary copy of X, and a Fréchet space V contains an isomorphic copy of Y, then (W, V) is a Bernstein pair with respect to $\{a_n\}$?

b) As for those Fréchet spaces which are not nuclear can we define Bernstein pairs and give examples?

To answer part a) perhaps using diagonal operators examining a theorem of BLT type for the space of bounded linear operators $\mathcal{L}(X, Y)$, under the condition $d_\nu > 0$, could be helpful.

For part b) recalling S. Bellenot's paper(s) such as *Local Reflexivity of Normed Spaces Operators and Fréchet Spaces* in [6], there he prove a non-uniformly convex Fréchet spaces has a conditional basic sequence, could be used.

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