Operating functions on modulation spaces

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Dedicated to Professor Katsuo Matsuoka on his 70th birthday

Abstract

In this paper, we survey recent progress with respect to the operating functions on modulation spaces, after we review about the operating functions in harmonic analysis.

§1. Introduction

Let $A(\mathbf{T})$ be the class of all continuous functions on the unit circle $\mathbf{T}$ with the absolutely convergent Fourier series. In 1932, Wiener[36] showed that $\frac{1}{f}$ is in $A(\mathbf{T})$ for every $f \in A(\mathbf{T})$ with $f(x) \neq 0$ for all $x \in \mathbf{T}$. This means that the composition $F \circ f$ is in $A(\mathbf{T})$ when $F(z) = \frac{1}{z}$ and $f \in A(\mathbf{T})$ with $f(x) \neq 0$ for all $x \in \mathbf{T}$. Lévy[27] gave an extension of this result which is called the Wiener-Lévy Theorem:

Theorem 1.1(Wiener-Lévy) Let D be a region in the complex plane, and $F(z)$ an analytic function on D. Then we have $F \circ f \in A(\mathbf{T})$ for every $f \in A(\mathbf{T})$ with $f(\mathbf{T}) \subset D$.

We remark that $A(\mathbf{T})$ is an algebra with pointwise multiplication. The study of N.Wiener developed into the studies of the commutative Banach algebras by I.M.Gelfand, D.A.Raikov, G.E.Shilov[10] and other mathematicians. The theory of those studies is called the Gelfand theory in modern analysis. A commutative Banach algebra is represented as a space of some continuous functions on a locally compact Hausdorff space by the Gelfand theory. In this sense, the study of operating functions enlightens the properties of commutative Banach algebras. The converse of the Wiener-Lévy Theorem was studied by some mathematicians, mainly, J.P.Kahane,Y.Katznelson, and etc.(cf.[16],[19],[30]). After all, Katznelson[18] solved by obtaining the following result, which is called the Katznelson Theorem:

Theorem 1.2(Katznelson) Let $I = [-1, 1]$, and F be a complex-valued function on I. If F operates on $A(\mathbf{T})$, F is extended to an analytic function on a neighborhood of I.

Definition 1.3 A complex-valued function F on $\mathbf{R}^2$ is said to be real analytic(resp. real entire) on $\mathbf{R}^2$ if for each $(s_0, t_0) \in \mathbf{R}^2$, F has a power series expansion

$$F(s,t) = \sum_{m,n=0}^{\infty} a_{mn}(s - s_0)^m (t - t_0)^n$$

$$(\text{resp. } F(s,t) = \sum_{m,n=0}^{\infty} a_{mn} s^m t^n)$$

which converges absolutely in a neighborhood of (s_0, t_0) (resp. in $\mathbf{R}^2$).

The operating functions of $A(\mathbf{T})$ defined by the domain $\mathbf{R}^2$ were decided by Helson, Kahane, Katznelson, Rudin[12], whose result is called the Helson-Kahane-Katznelson-Rudin Theorem:

Theorem 1.4(Helson-Kahane-Katznelson-Rudin) Let F be a complex-valued function on $\mathbf{R}^2$. If $F \circ f \in A(\mathbf{T})$ for every $f \in A(\mathbf{T})$, then F is real analytic on $\mathbf{R}^2$.

After that, there were investigated with respect to the operating functions on various function algebras related to Fourier series by many mathematicians from the 1960s to the 1970s. Here, we state some of those results:

Definition 1.5 Let K be a subset of the complex plane $\mathbf{C}$, and F a complex-valued function on K. Also let A and B be function spaces. It is said that F operates from A to B when $F \circ f \in B$ for every $f \in A$ with the range of $f \subset K$. When $A = B$, we say that F operates on A.

Theorem 1.6(Operating functions on the measure algebras)(cf.[30]) Let $M(\mathbf{T})$ be the measure algebra on $\mathbf{T}$ whose element is in the set of all bounded regular Borel measures on $\mathbf{T}$, and $M(\mathbf{T})^\wedge$ the set of all Fourier-Stieltjes transforms of $M(\mathbf{T})$. Then, a function F on the interval $[-1, 1]$ operates on $M(\mathbf{T})^\wedge$, if and only if F is extended to an entire function on $\mathbf{C}$.

For $1 \leq p < \infty$, let $M_p(\mathbf{T})$ be the set of all translation invariant bounded linear operators on $L^p(\mathbf{T})$, and $M_p(\mathbf{T})^\wedge$ the set of all Fourier-Stieltjes transforms of $M_p(\mathbf{T})$. Then $M_p(\mathbf{T})^\wedge$ is a commutative Banach algebra with pointwise multiplication and with the operator norm. Also it is known that

$$M_1(\mathbf{T}) = M(\mathbf{T}),\ M_2(\mathbf{T})^\wedge \sim \ell^\infty(\mathbf{Z}) \text{ (isomorphic)},$$

and

$$M_1(\mathbf{T}) \subset M_p(\mathbf{T}) \subset M_2(\mathbf{T}) \ (1 \le p < \infty),$$

where $\ell^\infty(\mathbf{Z})$ denotes the set of all bounded functions on the integer group $\mathbf{Z}$. When m_p denotes the closure of $L^1(\mathbf{T})$ in $M_p(\mathbf{T})$, Hörmander[13] showed the following result with respect to m_p:

Theorem 1.7(Hömander)

(1) The maximal ideal space of m_p is $\mathbf{Z}$. When F is a complex-valued function on $\mathbf{C}$ with $F(0) = 0$, F operates on $m_p^\wedge$, if F is analytic in a neighborhood of 0.

(2) Let $1 < p < q < 2$, and F be a complex-valued function on $\mathbf{C}$ with $F(0) = 0$, and $C_0M_p(\mathbf{T}) = \{T \in M_p(\mathbf{T}) | \hat{T}(n) \to 0 \text{ as } |n| \to \infty\}$. Then, if F is analytic in a neighborhood of 0, F operates from $C_0M_p(\mathbf{T})^\wedge$ to $M_q(\mathbf{T})^\wedge$.

Moreover, Hörmander[13] posed the problem "$C_oM_p(\mathbf{T}) = m_p$ for $p \neq 2$?". The following result is related to the problem:

Theorem 1.8(Igari[14]) Let $1 < p < 2$, and F be a complex-valued function on the interval $[-1, 1]$. F operates from $M_1(\mathbf{T})^\wedge$ to $M_p(\mathbf{T})^\wedge$, if and only if F is extended to an entire function on $\mathbf{C}$. Especially, F operates on $M_p(\mathbf{T})^\wedge$, if and only if F is extended to an entire function on $\mathbf{C}$.

The problem was solved by Figá-Talamanca-Gaudry[9], but Zafran[33] pointed out that their observation did not contribute to the operating functions of $C_0M_p(\mathbf{T})^\wedge$, and Zafran[34] obtained the characterization of the operating functions on $C_0M_p(\mathbf{T})^\wedge$ by applying the idea in Igari[15]. As a result, the Hömander problem was negatively solved:

Theorem 1.9(Zafran[34]) Let $1 < p < 2$, and F be a complex-valued function on $[-1, 1]$. F operates on $C_0M_p(\mathbf{T})^\wedge$, if and only if F is extended to an entire function on $\mathbf{C}$. By this, we have $m_p \neq C_0M_p(\mathbf{T})$ $(1 < p < 2)$.

The study of operating function on the function spaces related to Fourier series like those results in harmonic analysis had been developed in the 1970s.

By the way, Feichtinger[7] introduced the modulation spaces in 1983, and the basic theory was constructed by Feichtinger-Gröchenig[8], Gröchenig[11], and etc. By those studies, after that, it has been understood that the spaces are strongly related to nonlinear Schrödinger equations(NLS). Modulation spaces deeply associate between Schrödinger equations and harmonic analysis. In fact, we understand that the study of operating functions on modulation spaces are very useful for the study of Schrödinger equations which will be showed in §3.

This paper is organized by the following context. In §2, we state about the operating functions of $A^q_s(\mathbf{T})$ whose space is a generalization of $A(\mathbf{T})$. In §3, we state the some basic properties of modulation spaces with the definition of modulation spaces. Also we state about the usefulness of modulation spaces with respect to the study of NLS. In §4, it is stated that it was posed an open problem related to nonlinear Schrödinger equations with some initial data in [28], and was given the negative solution by Bhimani-Ratnakumar[4] investigated the operating functions on modulation spaces. Moreover, we pursue the development of their result. In §5, we state about the operating functions on the weighted modulation spaces $M^{p,q}_s(\mathbf{R})$, whose results are in Kobayashi-Sato[23].

The notation $E \lesssim F$ means $E \le cF$ for a some constant $c > 0$, whereas $A_1 \hookrightarrow A_2$ means the continuous embedding of the topological linear space A_1 into A_2. Also $A_1 \sim A_2$ means that A_1 is isomorphic to A_2 as a Banach algebra. For w in the n-dimensional Euclidean space $\mathbf{R}^n$, we denotes $(1+|w|^2)^{\frac{1}{2}}$ by $\langle w\rangle$, and for $1 \le q \le \infty$, q' the conjugate exponent of q.

§2. Operating functions on $A^q_s(\mathbf{T})$

Let $1 \le q < \infty,\ s \ge 0$, and $C(\mathbf{T}^n)$ be the set of all continuous functions on the n-dimensional torus. We also define $A^q_s(\mathbf{T}^n)$:

Definition 2.1

$$A^q_s(\mathbf{T}^n) = \{f \in C(\mathbf{T}^n) | f(x) = \sum_{k\in\mathbf{Z}^n} a_k e^{ikx},\ ||f||_{A^q_s} < \infty\},$$

where

$$||f||_{A^q_s} = (\sum_k (|a_k|\langle k\rangle^s)^q)^{\frac{1}{q}}.$$

In this section, we state about the operating functions of $A^q_s(\mathbf{T})$ on the 1-dimensional torus, according to Kobayashi-Sato[22]. As we said before, Y.Katznelson cleared the operating functions of $A^1_0(\mathbf{T}) = A(\mathbf{T})$. Moreover, he showed the operating functions of $A^1_s(\mathbf{T})$ $(s \ge 1)$ in [18].

Theorem 2.2(Katznelson) Let $s \geq 1$ and $\beta \geq 0$. Then, $F \in A^1_\beta(\mathbf{T})$ operates on all real-valued functions in $A^1_s(\mathbf{T})$, if and only if $\beta \geq s + \frac{1}{2}$.

For the case $0 < s < 1$, Leblanc[25,26] obtained the operating functions:

Theorem 2.3(Leblanc) Let $s > 0$ and $\beta \geq s + \frac{1}{2s} + \frac{5}{2}$. Then $F \in A^1_\beta(\mathbf{T})$ operates on all real-valued functions in $A^1_s(\mathbf{T})$.

Theorem 2.4(Leblanc) Let $0 < s < 1$ and $\beta \geq 0$. Then, we have the following:

(1) When $\beta > 1 + \frac{1}{2s}$, $F \in A^1_\beta(\mathbf{T})$ operates on all real-valued functions in $A^1_s(\mathbf{T})$.
(2) If $F \in A^1_\beta(\mathbf{T})$ operates on all real-valued functions in $A^1_s(\mathbf{T})$, we obtain $\beta \geq 1 + \frac{1}{2s}$.

Then, it is natural to ask the operating functions of $A^q_s(\mathbf{T})$ $(q > 1)$ in this direction. Here, we give some properties of this space:

(1) When $s > \frac{1}{q'}$, $A^q_s(\mathbf{T})$ is an algebra with pointwise multiplication.
(2) When $q_1 \leq q_2$ and $s_2 \leq s_1$, $A^{q_1}_{s_1}(\mathbf{T}) \hookrightarrow A^{q_2}_{s_2}(\mathbf{T})$.

Kobayashi-Sato[22] investigated the operating functions of $A^q_s(\mathbf{T})$, and obtained the following results:

Theorem 2.5 Let $1 < q < 2$, $s > \frac{1}{q'}$, and $\beta \geq s - \frac{1}{q'} + \frac{5}{2} + \frac{1}{2(s-\frac{1}{q'})}$. Then, $F \in A^1_\beta(\mathbf{T})$ operates on all real-valued functions in $A^q_s(\mathbf{T})$.

Theorem 2.6

(1) Let $1 < q < 2$, $s > 1+\frac{1}{q'}$, and $\beta \geq 0$. Then, $F \in A^1_\beta(\mathbf{T})$ operates on all real-valued functions in $A^q_s(\mathbf{T})$, if and only if $\beta \geq s + \frac{1}{q} - \frac{1}{2}$.
(2) Let $s > \frac{3}{2}$, and $\beta \geq 0$. Then, $F \in A^1_\beta(\mathbf{T})$ operates on all real-valued functions in $A^2_s(\mathbf{T})$, if and only if $\beta \geq s$.

Also we remark that every element of modulation spaces on $\mathbf{R}^n$ is locally in $A^q_s(\mathbf{T}^n)$(Bényi-Oh[3]), whose fact will be useful in §4.

§3. Modulation spaces and partial differential equations

In this section, first we define the modulation spaces on $\mathbf{R}^n$, and give the basic some properties. Moreover, we briefly state the importance of the modulation spaces for nonlinear Schrödinger equations, and also about an application of the operating functions of modulation spaces.

Now let $\mathcal{S}(\mathbf{R}^n)$ be the Schwartz space on $\mathbf{R}^n$, and $\mathcal{S}'(\mathbf{R}^n)$ the tempered distribution space. The Fourier transform of f is defined by

$$\widehat{f}(x) = \int f(\xi) e^{-ix\xi} d\xi.$$

Also we denote the set of all infinity partial differentiable functions on $\mathbf{R}^n$ with compact support by $C_c^\infty(\mathbf{R}^n)$. For $f \in \mathcal{S}'(\mathbf{R}^n)$ and $g \in \mathcal{S}(\mathbf{R}^n)$, $V_g f$ which is the Short-Time transform of f is defined by

$$\begin{aligned} V_g f(x,\xi) &= < f(x), g(t-x)e^{it\xi} > \\ &= \int f(t)\overline{g(t-x)} e^{-it\xi} dt. \end{aligned}$$

Definition 3.1(Modulation space) Let $1 \le p, q \le \infty$, and $s \ge 0$. For $0 \ne g \in \mathcal{S}(\mathbf{R}^n)$, we define

$$||f||_{M^{p,q}_{s,[g]}} = (\int (\int |V_g f(x,w)|^p dx)^{\frac{q}{p}} \langle w \rangle^{sq} dw)^{\frac{1}{q}},$$

and the modulation space $M^{p,q}_{s,[g]}(\mathbf{R}^n)$ is defined to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ with the norm $||f||_{M^{p,q}_{s,[g]}} < \infty$. If p or q equals ∞, $||f||_{M^{p,q}_{s,[g]}}$ corresponds to the essential supremum. Also we denote $M^{p,q}_{0,[g]}(\mathbf{R}^n)$ by $M^{p,q}_{[g]}(\mathbf{R}^n)$, but usually we omit g, and write $M^{p,q}_s(\mathbf{R}^n)$, $M^{p,q}(\mathbf{R}^n)$, respectively. Since $V_g\bar{f}(x,\xi) = \overline{V_g f(x,\xi)}$ for the complex conjugate function $\bar{f}$ of f, we have that $||\bar{f}||_{M^{p,q}_s} \le ||f||_{M^{p,q}_s}$ and $||\mathrm{Im}\ f||_{M^{p,q}_s} \le ||f||_{M^{p,q}_s}$. Here, we give some basic properties of the modulation spaces(cf. Gröchenig[11]).

Proposition 3.2 Let $1 \le p, q, p_i, q_i \le \infty$ $(i = 1, 2)$. Then, we have the following results:

(1) $M^{p,q}_s(\mathbf{R}^n)$ is a Banach space.

(2) $\mathcal{S}(\mathbf{R}^n) \subset_\to M^{p,q}(\mathbf{R}^n) \subset_\to \mathcal{S}'(\mathbf{R}^n)$.

(3) $M^{p,q_1}(\mathbf{R}^n) \subset_\to L^p(\mathbf{R}^n) \subset_\to M^{p,q_2}(\mathbf{R}^n)$ $(q_1 \le \min\{p, p'\},\ q_2 \ge \max\{p, p\prime\})$.

(4) when $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$ and $\frac{1}{q_1}+\frac{1}{q_2}=1+\frac{1}{q}$,

$$||fg||_{M^{p,q}} \lesssim ||f||_{M^{p_1,q_1}}||g||_{M^{p_2,q_2}} \ (f \in M^{p_1,q_1}(\mathbf{R}^n),\ g \in M^{p_2,q_2}(\mathbf{R}^n)).$$

The modulation spaces $M_s^{p,q}(\mathbf{R}^n)$ are useful for the nonlinear Schrödinger equations:

$$(NLS) \quad i\frac{\partial}{\partial t}u(x,t)+\Delta_x u(x,t)=F(u(x,t)), \quad u(x,0)=u_0(x),$$

where $\Delta_x=\sum_{n=1}^{n}\frac{\partial^2}{\partial x_j^2},\ (x,t)\in \mathbf{R}^n\times\mathbf{R},\ i=\sqrt{-1}$, and $u_0,\ F$ are complex-valued functions on $\mathbf{R}^n$, or $\mathbf{C}$, respectively. The modern approach to study the well-posedness of dispersive equations is done by applying the operators for the associated dispersive equations(cf.[20],[31]). Here, we have an interesting subclass of L^p which is the modulation space $M^{p,1}(\mathbf{R}^n)$ for $1\le p\le\infty$. In fact, the modulation spaces have been studied by many mathematician, and there were obtained interesting properties, that are known to fail on Lebesgue spaces. For example, the multipliers $e^{it4\pi^2|\xi|^2}$ corresponding to the Schrödinger propagator $e^{-it\Delta}$, provides a bounded Fourier multiplier operator on $L^p(\mathbf{R}^n)$ only for $p=2$. This is a classical theorem of Hörmander[13](cf. Lebedev-Olveskii[24]), where as all multipliers $e^{it|\xi|^a}$ $(0\le a\le 2)$ defines bounded Fourier multipliers operators on $M_s^{p,q}(\mathbf{R}^n)$ $1\le p,q\le\infty,\ s\ge 0$, as shown by Bényi-Gröchenig-Okoudjou-Rogers[1](cf. [2],[35]).

Now the local well-posedness of NLS in $M^{2,1}(\mathbf{R}^n)$ is a result of Wang-Zhao-Guo[35] with non-linearities that include $F_k(u)=\lambda|u|^{2k}u$ $(k\in\mathbf{N},\ \lambda\in\mathbf{R})$ etc.(cf.[2]). In this case, one of the key points is that, the nonlinearities terms map the modulation space to itself. In fact, the above results are based on the fact that $M_s^{p,1}(\mathbf{R}^n)$ is a function algebra under pointwise multiplication:

$$||fg||_{M_s^{p,1}} \lesssim ||f||_{M_s^{p,1}}||g||_{M_s^{p,1}}.$$

Therefore, if $\alpha=2k,\ |u|^\alpha u=u^{k+1}\bar{u}^k$ and hence $|||u|^\alpha u||_{M_s^{p,1}}\lesssim ||u||_{M_s^{p,1}}^{2k+1}$. Then it is natural to ask how far we can investigate including more general nonlinear terms in these dispersive equations on modulation spaces. In the context, Ruzhansky-Sugimoto-Wang[28] posed an open problem, and Bhimani-Ratnakumar[4] solved it negatively. In §4, we state their results with the related results.

§4. Operating functions on $M^{p,1}(\mathbf{R}^n)$

In this section, we state with respect to the open problem of Ruzhansky-Sugimoto-Wang[28], which was negatively solved by Bhimani-Ratnakumar[4] by using the operating functions in harmonic analysis, and the related results were obtained by Kobayashi-Sato[21] and Bhimani[5].

First we raise the problem in [28]:

Problem 4.1(Ruzhansky-Sugimoto-Wang) Let $\alpha \in (0, \infty)\backslash 2\mathbf{N}$. Does

$$||f|f|^{\alpha}||_{M^{p,1}} \lesssim ||f||_{M^{p,1}}^{\alpha+1}$$

hold for all $u \in M^{p,1}(\mathbf{R}^n)$?

Definition 4.2 Let F be a complex-valued function on $\mathbf{R}^2$. If $F(\mathrm{Re}\ f, \mathrm{Im}\ f) \in M^{p,1}(\mathbf{R}^n)$ for every $f \in M^{p,1}(\mathbf{R}^n)$, then we say that F operates on $M^{p,1}(\mathbf{R}^n)$.

It is known that $f \in M^{p,1}(\mathbf{R}^n)$ is locally in $A(\mathbf{T}^n)$. In fact, we have the following result in [3]:

Proposition 4.3(Bényi-Oh) Let $f \in M^{p,1}(\mathbf{R}^n)$, $1 \le p < \infty$ and ϕ a smooth function supported on $[0, 2\pi)^n$. Then $\phi f \in A(\mathbf{T}^n)$ and satisfies the inequality

$$||\phi f||_{A(\mathbf{T}^n)} \lesssim ||f||_{M^{p,1}}.$$

Bhimani-Ratnakumar[4] obtained the following result by applying this result and the Helson-Kahane-Katznelson-Rudin Theorem in §2.

Theorem 4.4(Bhimani-Ratnakumar) Let $1 \le p < \infty$ and F be a complex-valued function on $\mathbf{R}^2$. If F operates on $M^{p,1}(\mathbf{R}^n)$, then F is a real analytic function on $\mathbf{R}^2$ with $F(0) = 0$. Conversely, if F is a real analytic function on $\mathbf{R}^2$ with $F(0) = 0$, then F operates on $M^{1,1}(\mathbf{R}^n)$.

This result gives the negative answer of the problem 4.1:

Corollary 4.5 There exists $f \in M^{p,1}(\mathbf{R}^n)$ such that $f|f|^{\alpha} \notin M^{p,1}(\mathbf{R}^n)$ for any $\alpha \in (0, \infty)\backslash 2\mathbf{N}$.

However, they did not have the sufficient condition of the operating function on $M^{p,1}(\mathbf{R}^n)$ $(p > 1)$, and posed the open problem in [4], namely, "F operates on $M^{p,1}(\mathbf{R}^n)$ $(p > 1)$, if F is a real analytic function on $\mathbf{R}^2$?"

Kobayashi-Sato[21] solved the problem in 1-dimensional case, and Bhimani[5] in n-dimensional case by the similar idea in Kobayashi-Sato[21]:

Theorem 4.6(Kobayashi-Sato, Bhimani) Let $1 \le p < \infty$ and F be a complex-valued function on $\mathbf{R}^2$ with $F(0) = 0$. Then F operates on $M^{p,1}(\mathbf{R}^n)$, if and only if F is a real analytic function on $\mathbf{R}^2$.

Bhimani-Ratnakumar[4] gave some applications to nonlinear Schrödinger equations by applying that real entire functions operate on the space of some Banach algebras in $M_s^{p,q}(\mathbf{R}^n)$, which are related to the estimates of some operators associated to NLS.

§5. Operating functions on the weighted modulation spaces

In this section, we state about the operating functions of the weighted modulation spaces on the 1-dimensional Euclidean space, according to Kobayashi-Sato[23].

As the modulation spaces are stated the usefulness for the study of NLS in §4, if we are interested in the well-posedness theory of nonlinear PDEs with initial data in a modulation space, we need to understand the behavior of the modulation space under the nonlinearities such as $|f|^\alpha$ and $f|f|$ for $f \in M_s^{p,q}$. These types of issues have been studied by many mathematician using a large variety of methods. For instance, Sugimoto-Tomita-Wang[32] obtained the following result:

Theorem 5.1(Sugimoto-Tomita-Wang) Let $s > \frac{1}{2}$, and let $\alpha > 0$, $\alpha \notin \mathbf{Z}$. Assume that $f \in M_s^{p,2}(\mathbf{R})$ is real-valued. Then $|f|^\alpha$, $f|f|^{\alpha-1} \in M_s^{p,2}(\mathbf{R})$ if $[\alpha] > [s]+2$ and $1 \leq p \leq 2([\alpha]-[s]-1)$, where $[\alpha]$ denotes the integer part of $\alpha > 0$.

Theorem 5.1 raises the expectation that similar results may work with less restrictive windows. Ruzhansky-Sugimoto-Wang[28] proposed an open problem with the algebraic properties of $M_0^{p,1}$ needed to study NLS and Klein-Gordon equations with general power type negatively of the $f|f|^\alpha$. However, Bhimani-Ratnakumar[4] solved it negatively with ideas related to the operating functions, as we said in §4. From Theorem 4.1 in §4, we see that there are functions $f \in M_0^{p,1}(\mathbf{R})$ for which $|f|$ and $f|f|^{2k+1}$ do not belong to $M_0^{p,1}$. Considering the result of Bhimani-Ratnakmar[4] and Theorem 5.1, it is natural to find other function F such that $F(f) \in M_s^{p,q}$ for all $f \in M_s^{p,q}$. In the frame work of C^∞-function, Kato-Sugimoto-Tomita[17] studied the operating functions on $M_s^{p,q}$ for $q \geq \frac{4}{3}$ and obtained the following result:

Theorem 5.2(Kato-Sugimoto-Tomita) Let $1 \leq p < \infty$, $\frac{4}{3} \leq q < \infty$ and $s > \frac{1}{q'}$, where q' denotes conjugate exponent of q. If $f : \mathbf{R} \to \mathbf{R}$, $f \in M_s^{p,q}(\mathbf{R})$, $F \in C^\infty(\mathbf{R})$ and $F(0) = 0$, then $F(f) \in M_s^{p,q}(\mathbf{R})$.

On the other hand, in the frame work of measure on $\mathbf{R}$, Reich-Sickel[29] studied the operating functions on $M_s^{p,q}$ and obtained the following result:

Theorem 5.3(Reich-Sickel[29;Theorem 4.3]) Let $1 < p < \infty,\ 1 \leq q \leq \infty$ and $s > \frac{1}{q'}$. let μ be a measure on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} (1+|\xi|)^{1+(s+\frac{1}{q})(1+\frac{q'}{sq'-1})} d|\mu| < \infty$$

and $\mu(\mathbf{R}) = 0$. Furthermore, assume that the function F is the Fourier transform of μ. Then $F(f) \in M_s^{p,q}(\mathbf{R})$ for all $f \in M_s^{p,q}(\mathbf{R})$.

As pointed out in Reich-Sickel[29], we note that as the case of Sobolev space $H^s = M_s^{2,2}$ shows(cf.[6]), the conditions in Theorem 5.3 are not very closed to the necessary conditions. Thus there is a certain gap. Moreover, as far as we know, it seems that little is known above the necessary conditions to characterize the operating functions on $M_s^{p,q}$, except for the case $p = q = 2$. Kobayashi-Sato[23] has obtained the preciseness of the operating function on the weighted modulation spaces, which corresponds to the discrete version of Theorem 5.3. More precisely, we give necessary and sufficient conditions for all functions $f \in A_\beta^1(\mathbf{T})$ to operate on all real-valued functions in $M_s^{p,q}(\mathbf{R})$:

Theorem 5.4(Necessary conditions) Let $1 \leq p < \infty,\ 1 < q \leq 2,\ s > \frac{1}{q'}$ and $\beta \geq 0$. If all functions $F \in A_\beta^1(\mathbf{T})$ with $F(0) = 0$ operates on all real-valued functions in $M_s^{p,q}(\mathbf{R})$, then $\beta \geq s + \frac{1}{q} - \frac{1}{2}$.

Theorem 5.5(Sufficient conditions) Let $1 < q < 2 \leq p < \infty$.

(1) Suppose $s = \frac{1}{q'} + \frac{1}{2}$ or $s = \frac{1}{q'} + \frac{3}{2}$. If $\beta > s + \frac{1}{q} - \frac{1}{2}$ and $F \in A_\beta^1(\mathbf{T})$ with $F(0) = 0$, then $F(f) \in M_s^{p,q}(\mathbf{R})$ for all real-valued functions $f \in M_s^{p,q}(\mathbf{R}) \cap M_2^{p,2}(\mathbf{R}) \cap M_1^{2,2}(\mathbf{R})$.

(2) Suppose $s > \frac{1}{q'} + \frac{1}{2},\ s \neq \frac{1}{q'} + \frac{3}{2}$. If $\beta \geq s$ and $F \in A_\beta^1(\mathbf{T})$ with $F(0) = 0$, then $F(f) \in M_s^{p,q}(\mathbf{R})$ for all real-valued functions $f \in M_s^{p,q}(\mathbf{R}) \cap M_2^{p,2}(\mathbf{R}) \cap M_1^{2,2}(\mathbf{R})$.

(3) Suppose $s \geq 1$. If $\beta \geq s$ and $F \in A_\beta^1(\mathbf{T})$ with $F(0) = 0$, then $F(f) \in M_s^{p,2}(\mathbf{R})$ for all real-valued functions $f \in M_s^{p,2}(\mathbf{R}) \cap M_2^{p,2}(\mathbf{R}) \cap M_1^{2,2}(\mathbf{R})$.

Corollary 5.6 Let $1 < q < 2 < s$ or $q = s = 2$, and $\beta \geq 0$. Then all functions $F \in A_\beta^1(\mathbf{T})$ with $F(0) = 0$ operate on all real-valued functions $f \in M_s^{2,q}(\mathbf{R})$, if and only if $\beta \geq s + \frac{1}{q} - \frac{1}{2}$.

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