Uncertainty Relations for Generalized Quasi-Metric Adjusted Skew Information and Their Applications

Kenjiro Yanagi

Abstract

We investigate some uncertainty relations for generalized quasi-metric adjusted skew informations. We give generalized Heisenbrg type or generalized Schrödinger type uncertainty relations. And we obtain several important norm inequalities which refine the triangle inequality and Hlawka inequality and so on in order to formulate sum types of uncertainty relations for N not necessarily hermitian quantum mechanical observables. As applications we have some norm inequalities. At last we state uncertainty relations for quantum channels.

1. Introduction

Let $M_n(\mathbb{C})$ be a set of all $n \times n$ complex matrices, $M_{n,sa}(\mathbb{C})$ be a set of all $n \times n$ self-adjoint matrices, $M_{n,+}(\mathbb{C})$ be a set of all $n \times n$ positive semi-definite matrices and $M_{n,+,1}(\mathbb{C})$ be a set of all $n \times n$ density matrices. That is $M_{n,+,1}(\mathbb{C}) = \{\rho \in M_n(\mathbb{C}) | Tr[\rho] = 1, \rho > 0\}$. Let $\langle A, B \rangle = Tr[A^* B]$ be a Hilbert-Schmidt scalar product. For $\rho \in M_{n,+,1}(\mathbb{C})$ and $A, B \in M_{n,sa}(\mathbb{C})$, an expectation of A under physical state ρ is given by $E_\rho(A) = Tr[\rho A]$ and a variance is given by $V_\rho(A) = Tr[\rho A_0^2]$ where $A_0 = A - Tr[\rho A]I$. For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$, the famous Heisenberg uncertainty relation ([16]) is given by

$$V_\rho(A) \cdot V_\rho(B) \geq \frac{1}{4} |Tr[\rho[A, B]]|^2,$$

where $[A, B] = AB - BA$. And also the Schrödinger uncertainty relation ([24]) is given by

$$V_\rho(A) \cdot V_\rho(B) - |\mathrm{Re}\{Tr[\rho A_0 B_0]\}|^2 \geq \frac{1}{4} Tr[\rho[A, B]]|^2,$$

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which is a refinement of Heisenberg uncertainty relation. The Wigner-Yanase skew information is defined by

$$I_\rho(A) = \frac{1}{2}Tr[(i[\rho^{1/2}, A_0])^2] = Tr[\rho A_0^2] - Tr[\rho^{1/2}A_0\rho^{1/2}A_0],$$

which is smaller than $V_\rho(A)$. And the related value is defined by

$$J_\rho(A) = \frac{1}{2}Tr[\rho\{A_0, B_0\}^2] = Tr[\rho A_0^2] + Tr[\rho^{1/2}A_0\rho^{1/2}A_0],$$

where $\{A, B\} = AB + BA$. Now we define

$$U_\rho(A) = \sqrt{I_\rho(A) \cdot J_\rho(A)}.$$

It is clear that $0 \le I_\rho(A) \le U_\rho(A) \le V_\rho(A)$. For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$, the uncertainty relation for Wigner-Yanase skew information ([21]) is given by

$$U_\rho(A) \cdot U_\rho(B) \ge \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

After then the Wigner-Yanase-Dyson skew information is defined by

$$I_{\rho,\alpha}(A) = \frac{1}{2}Tr[(i[\rho^\alpha, A_0])(i[\rho^{1-\alpha}, A_0])] = Tr[\rho A_0^2] - Tr[\rho^\alpha A_0\rho^{1-\alpha}A_0],$$

where $0 \le \alpha \le 1$. And the related value is defined by

$$J_{\rho,\alpha}(A) = \frac{1}{2}Tr[\{\rho^\alpha, A_0\}\{\rho^{1-\alpha}, A_0\}] = Tr[\rho A_0^2] + Tr[\rho^\alpha A_0\rho^{1-\alpha}A_0].$$

Now we define

$$U_{\rho,\alpha}(A) = \sqrt{I_{\rho,\alpha}(A) \cdot J_{\rho,\alpha}(A)}.$$

It is clear that

$$0 \le I_{\rho,\alpha}(A) \le I_\rho(A) \le U_\rho(A),\ 0 \le I_{\rho,\alpha}(A) \le U_{\rho,\alpha}(A) \le U_\rho(A).$$

For $A, B \in M_{n,sa}(\mathbb{C})$, $\rho \in M_{n,+,1}(\mathbb{C})$, the uncertainty relation for Wigner-Yanase-Dyson skew information ([27]) is given by

$$U_{\rho,\alpha}(A) \cdot U_{\rho,\alpha}(B) \ge \alpha(1-\alpha)|Tr[\rho[A, B]]|^2.$$

Hansen ([15]) defined the following metric adjusted skew information. Let $\mathfrak{F}_{op} = \{f : (0, \infty) \to (0, \infty) | f(1) = 1,\ xf(x^{-1}) = f(x),\ f \text{ is operator monotone}\}$ and let $\mathfrak{F}_{op}^r = \{f \in \mathfrak{F}_{op} | f(0) \ne 0\}$ and $\mathfrak{F}_{op}^n = \{f \in \mathfrak{F}_{op} | f(0) = 0\}$, where $f(0) = \lim_{x\to 0} f(x)$. Then it is clear that $\mathfrak{F}_{op} = \mathfrak{F}_{op}^r \cup \mathfrak{F}_{op}^n$. We define

$$\tilde{f}(x) = \frac{1}{2}\left[(x+1) - (x-1)^2\frac{f(0)}{f(x)}\right],\ \ x > 0,\ f \in \mathfrak{F}^r.$$

The correspondence $f \to \tilde{f}$ is a bijection between $\mathfrak{F}_{op}^r$ and $\mathfrak{F}_{op}^n$. The examples of $\mathfrak{F}_{op}$ are as follows.

$$f_{RLD}(x) = \frac{2x}{x+1},\ \ f_{BKN}(x) = \frac{x-1}{\log x},\ \ f_{SLD}(x) = \frac{x+1}{2},$$

$$\tilde{f}_{SLD}(x) = \frac{2x}{x+1}, \quad f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \quad \tilde{f}_{WY}(x) = \sqrt{x},$$

$$f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)}, \quad \alpha \in (0,1),$$

$$\tilde{f}_{WYD}(x) = \frac{1}{2}\{x + 1 - (x^\alpha - 1)(x^{1-\alpha} - 1)\}.$$

Then there are the following relationships among the above examples.

$$\frac{2x}{x+1} < \sqrt{x} < \frac{x-1}{\log x} < f_{WYD} < \left(\frac{\sqrt{x}+1}{2}\right)^2 < \frac{x+1}{2} \quad (x \neq 1)$$

In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathfrak{F}_{op}$ by the formula

$$m_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $A, B \in M_{n,+}(\mathbb{C})$. Now the monotone metrics(also said quantum Fisher informations) is defined by

$$\langle A, B \rangle_f = Tr[A \cdot m_f(L_\rho, R_\rho)^{-1}(B)],$$

where $L_\rho(A) = \rho A, R_\rho(A) = A\rho, A, B \in M_{n,sa}(\mathbb{C})$. The metric adjusted skew information $I_\rho^f(A)$ is defined as follows. Let

$$Corr_\rho^f(A,B) = Tr[\rho A_0 B_0] - Tr[A_0 m_{\tilde{f}}(L_\rho, R_\rho) B_0],$$

$$I_\rho^f(A) = Corr_\rho^f(A,A) = Tr[\rho A_0^2] - Tr[A_0 m_{\tilde{f}}(L_\rho, R_\rho) A_0].$$

And the related value is defined by

$$J_\rho^f(A) = Tr[\rho A_0^2] + Tr[A_0 m_{\tilde{f}}(L_\rho, R_\rho) A_0].$$

Now we define

$$U_\rho^f(A) = \sqrt{I_\rho^f(A) \cdot J_\rho^f(A)}.$$

It is clear that

$$0 \leq I_\rho^f(A) \leq U_\rho^f(A) \leq V_\rho(A).$$

For $A, B \in M_{n,sa}(\mathbb{C}), \rho \in M_{n,+,1}(\mathbb{C})$ and $f \in \mathfrak{F}^r$, the Schrödinger type uncertainty relation for metric adjusted skew information ([29]) is given by

$$I_\rho^f(A) \cdot I_\rho^f(B) \geq |Corr_\rho^f(A,B)|^2.$$

On the other hand under the condition $\frac{x+1}{2} + \tilde{f}(x) \geq 2f(x)$, the Heisenbeg type uncertainty relation for metric adjusted skew information ([29]) is given by

(1) $U_\rho^f(A) \cdot U_\rho^f(B) \geq f(0)|Tr[\rho[A,B]]|^2$.

(2) $U_\rho^f(A)\cdot U_\rho^f(B)\geq 4f(0)|Corr_\rho^f(A,B)|^2$.

Furthermore we define the generalized metric adjusted skew information as follows. Let $g,f\in\mathfrak{F}_{op}^r$ satisfy

$$g(x)\geq k\frac{(x-1)^2}{f(x)} \tag{1.1}$$

for some $k>0$. We define

$$\Delta_g^f(x)=g(x)-k\frac{(x-1)^2}{f(x)}\in\mathfrak{F}_{op}.$$

When $f(x)>0$ on $(0,\infty)$, the followings are equaivalent ([17]).

(1) $f(x)$ is operator monotone,

(2) $\dfrac{x-1}{f(x)}$ is operator monotone,

(3) $(x-1)f(x)$ is operator convex,

(4) $\dfrac{(x-1)^2}{f(x)}$ is operator convex.

Then since $f(x)>0$ on $(0,\infty)$, $-k\frac{(x-1)^2}{f(x)}$ is operator concave. And also since $g(x)$ is operator concave, $\Delta_g^f(x)$ is operator concave. Since $\Delta_g^f(x)>0$ on $(0,\infty)$, $\Delta_g^f(x)$ is operator monotone. The generalized metric adjusted skew iformation $I_\rho^{(g,f)}(A)$ is defined as follows. Let

$$\begin{aligned}Corr_\rho^{(g,f)}(A,B) &= k\langle i[\rho,A_0],i[\rho,B_0]\rangle_f\\ &= Tr[A_0m_g(L_\rho,R_\rho)B_0]-Tr[A_0m_{\Delta_g^f}(L_\rho,R_\rho)B_0],\end{aligned}$$

$$\begin{aligned}I_\rho^{(g,f)}(A) &= Corr_\rho^{(g,f)}(A,A)\\ &= Tr[A_0m_g(L_\rho,R_\rho)A_0]-Tr[A_0m_{\Delta_g^f}(L_\rho,R_\rho)A_0].\end{aligned}$$

And the related value is defined by

$$J_\rho^{(g,f)}(A)=Tr[A_0m_g(L_\rho,R_\rho)A_0]+Tr[A_0m_{\Delta_g^f}(L_\rho,R_\rho)A_0].$$

Now we define

$$U_\rho^{(g,f)}(A)=\sqrt{I_\rho^{(g,f)}(A)\cdot J_\rho^{(g,f)}(A)}.$$

For $A,B\in M_{n,sa}(\mathbb{C})$, $\rho\in M_{n,+,1}(\mathbb{C})$, the Schrödinger type uncertainty relation for generalized metric adjusted skew information ([31]) is given by

$$I_\rho^{(g,f)}(A)\cdot I_\rho^{(g,f)}(B)\geq|Corr_\rho^{(g,f)}(A,B)|^2.$$

On the other hand under the condition $g(x) + \Delta_g^f(x) \geq \ell f(x)$ for some $\ell > 0$, the Heisenberg type uncertainty relation for generalized metric adjusted skew information ([31]) is given by

$$U_\rho^{(g,f)}(A) \cdot U_\rho^{(g,f)}(B) \geq k\ell |Tr[\rho[A,B]]|^2.$$

In this paper we give the Schrödinger/Heisenberg type uncertainty relation for generalized quasi-metric adjusted skew information. In section 2, we define generalized quasi-metric adjusted skew information and state the theorem. And as application we give the new inequalities for fidelity and trace distance. In section 3, we propose the sum type of uncertainty relation for generalized quasi-metric adjusted skew information by extending the norm inequalities. In section 4, we state the uncertainty relations for quantum channels.

2. Uncertainty Relation for Generalized Quasi-Metric Adjusted Skew Information

Definition 2.1. For $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$, we define the following quantities:

(1) $\Gamma_{A,B}^{(g,f)}(X,Y) = k\langle (L_A - R_B)X, (L_A - R_B)Y\rangle_f$

$$= kTr[X^*(L_A - R_B)m_f(L_A, R_B)^{-1}(L_A - R_B)Y]$$

$$= Tr[X^* m_g(L_A, R_B)Y] - Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y],$$

(2) $I_{A,B}^{(g,f)}(X) = \Gamma_{A,B}^{(g,f)}(X,X)$,

(3) $\Psi_{A,B}^{(g,f)}(X,Y) = Tr[X^* m_g(L_A, R_B)Y] + Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y]$,

(4) $J_{A,B}^{(g,f)}(X) = \Psi_{A,B}^{(g,f)}(X,X)$,

(5) $U_{A,B}^{(g,f)}(X) = \sqrt{I_{A,B}^{(g,f)}(X) \cdot J_{A,B}^{(g,f)}(X)}$.

The quantity $I_{A,B}^{(g,f)}(X)$ and $\Gamma_{A,B}^{(g,f)}(X,Y)$ are said generalized quasi-metric adjusted skew information and generalized quasi-metric adjusted correlation measure, respectively.

Theorem 2.2 (Schrödinger type, [35]). *For $f \in \mathfrak{F}_{op}^r$, it holds*

$$I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \geq |\Gamma_{A,B}^{(g,f)}(X,Y)|^2 \geq \frac{1}{16}\left(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)\right)^2,$$

where $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$.

Theorem 2.3 (Heisenberg type, [35]). *For $f \in \mathfrak{F}_{op}^r$, if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \tag{2.1}$$

for some $\ell > 0$, then

(1) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq k\ell |Tr[X^*|L_A - R_B|Y]|^2$,
(2) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq \dfrac{f(0)^2\ell}{k} |\Gamma_{A,B}^{(g,f)}(X,Y)|^2$,

where $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$.

We assume that

$$g(x) = \frac{x+1}{2},\ f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},\ k = \frac{f(0)}{2},\ \ell = 2.$$

Then since (1.1), (2.1) are satisfied, we have the following trace inequality by putting $X = Y = I$ in Theorem 2.3.

$$\begin{aligned} & \alpha(1-\alpha)(Tr[|L_A - R_B|I])^2 \\ \leq\ & \left(\frac{1}{2}Tr[A+B]\right)^2 - \left(\frac{1}{2}Tr[A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha]\right)^2. \end{aligned}$$

Since

$$Tr[|L_A - R_B|I] = \sum_{i=1}^n \sum_{j=1}^n |\lambda_i - \mu_j| |\langle \phi_i | \psi_j \rangle|^2,$$

we have

$$\begin{aligned} & 2Tr[A^\alpha B^{1-\alpha}] - Tr[A + B - |L_A - R_B|I] \\ =\ & \sum_{i=1}^n \sum_{j=1}^n \{2\lambda_i^\alpha \mu_j^{1-\alpha} - (\lambda_i + \mu_j - |\lambda_i - \mu_j|)\} |\langle \phi_i | \psi_j \rangle|^2 \geq 0. \end{aligned}$$

Then we give the following trace inequality.

$$\frac{1}{2}Tr[A + B - |L_A - R_B|I] \leq Tr[A^\alpha B^{1-\alpha}].$$

Theorem 2.4 ([33]). *We have the following:*

$$\begin{aligned} & \frac{1}{2}Tr[A + B - |L_A - R_B|I] \leq \inf_{0 \leq \alpha \leq 1} Tr[A^{1-\alpha}B^\alpha] \\ \leq\ & Tr[A^{1/2}B^{1/2}] \leq \frac{1}{2}Tr[A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha] \\ \leq\ & \sqrt{\left(\frac{1}{2}Tr[A+B]\right)^2 - \alpha(1-\alpha)\left(Tr[|L_A - R_B|I]\right)^2}. \end{aligned}$$

Remark 2.5. We have three remarks.

(1) There is no relationship between $Tr[|L_A - R_B|I]$ and $Tr[|A-B|]$. Because if

$$A = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}, \ B = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix},$$

then $\mathrm{Tr}(|L_A - R_B|I) = 3, \ \mathrm{Tr}(|A-B|) = \sqrt{10}$.
On the other hand if

$$A = \begin{pmatrix} \frac{13}{2} & \frac{7}{2} \\ \frac{7}{2} & \frac{13}{2} \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix},$$

then $\mathrm{Tr}(|L_A - R_B|I) = 8, \ \mathrm{Tr}(|A-B|) = \sqrt{58}$.

(2) Theorem 2.4 is a generalization of the following result by Powers-Störmer [23] and Audenaert et [1]:

$$\begin{aligned} \frac{1}{2}Tr[A+B-|A-B|] &\leq \inf_{0\leq\alpha\leq 1} Tr[A^{1-\alpha}B^{\alpha}] \\ \leq \ Tr[A^{1/2}B^{1/2}] &\leq \sqrt{\left(\frac{1}{2}Tr[A+B]\right)^2 - \left(\frac{1}{2}Tr[|A-B|]\right)^2}. \end{aligned}$$

(3) When $A, B \in M_{2,+,1}(\mathbb{C})$, we can prove

$$Tr[|L_A - R_B|I] \leq Tr[|A-B|].$$

When $n \geq 3$, it is a conjecture.

Theorem 2.6 ([33])**.**

$$Tr[|A^{1/2}B^{1/2}|] \geq \frac{1}{1+\sqrt{\lambda_0}}Tr[A] + \frac{\sqrt{\lambda_0}}{1+\sqrt{\lambda_0}}\left(\frac{1}{2}Tr[A+B-|A-B|]\right),$$

where λ_0 is the largest eigenvalue of $B^{-1/2}AB^{-1/2}$.

3. Sum Type Uncertainty Relation

Theorem 3.1 ([36])**.** *For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, we have the following.*

(1) $I_{A,B}^{(g,f)}(X) + I_{A,B}^{(g,f)}(Y) \geq \frac{1}{2}\max\{I_{A,B}^{(g,f)}(X+Y), I_{A,B}^{(g,f)}(X-Y)\}$.

(2) $\sqrt{I_{A,B}^{(g,f)}(X)} + \sqrt{I_{A,B}^{(g,f)}(Y)} \geq \max\{\sqrt{I_{A,B}^{(g,f)}(X+Y)}, \sqrt{I_{A.B}^{(g,f)}(X-Y)}\}$.

Theorem 3.2 ([36]). *For* $\{X_i\}_{i=1}^N, \{Y_j\}_{j=1}^N \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, *we assume that* $X_i^*|L_A - R_B|Y_j = \delta_{ij}C$ *and Condition (2.1) is satisfied. Then (1) and (2) hold.*

(1) $$\left(\sum_{i=1}^N U_{A,B}^{(g,f)}(X_i)\right)\left(\sum_{j=1}^N U_{A,B}^{(g,f)}(Y_j)\right) \geq Nk\ell|Tr[C]|^2.$$

(2) $$\left(\sum_{i=1}^N \sqrt{U_{A,B}^{(g,f)}(X_i)}\right)\left(\sum_{j=1}^N \sqrt{U_{A,B}^{(g,f)}(Y_j)}\right) \geq N\sqrt{k\ell}|Tr[C]|.$$

Theorem 3.3 ([36]). *For* $\{X_i\}_{i=1}^N \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, *we put*

$$X^+ = I_{A,B}^{(g,f)}(X_i + X_j),\ X^- = I_{A,B}^{(g,f)}(X_i - X_j),$$

$$Y = I_{A,B}^{(g,f)}(X_i),\ Z = I_{A,B}^{(g,f)}(\sum_{i=1}^N X_i).$$

Then (1), (2) and (3) hold.

(1) $$\sum_{i=1}^N I_{A,B}^{(g,f)}(X_i)$$

$$\geq \frac{1}{N-2}\sum_{1\leq i<j\leq N} I_{A,B}^{(g,f)}(X_i+X_j) - \frac{1}{(N-1)^2(N-2)}\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)}\right)^2.$$

(2) $$\sum_{i=1}^N \sqrt{I_{A,B}^{(g,f)}(X_i)} \geq \frac{1}{N-2}\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)} - \sqrt{I_{A,B}^{(g,f)}\left(\sum_{i=1}^N X_i\right)}\right)$$

$$\geq \frac{1}{N-1}\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)} \geq \max\left\{\frac{1}{N-2}\left(\sum_{i<j}\sqrt{X^+} - \sum_{i=1}^N\sqrt{Y}\right), \sqrt{Z}\right\}.$$

(3) $$\frac{1}{N(N-1)^2}\left\{\left(\sum_{i<j}\sqrt{X^+}\right)^2 + \left(\sum_{i<j}\sqrt{X^-}\right)^2\right\}$$

$$\leq \sum_{i=1}^N I_{A,B}^{(g,f)}(X_i) \leq \frac{1}{N}\sum_{i<j}X^- + \frac{1}{N(N-1)^2}\left(\sum_{i<j}\sqrt{X^+}\right)^2.$$

Lemma 3.4. *Let $\|\cdot\|$ be the Hilbert-Schmidt norm on $M_n(\mathbb{C})$. For $\{A_i\}_{i=1}^N \subset M_n(\mathbb{C})$, we put*

$$U = \sum_{i=1}^N \|A_i\|, \; W = \|\sum_{i=1}^N A_i\|$$

$$V^+ = \frac{1}{N-1}\sum_{i<j}\|A_i + A_j\|, \; V^- = \frac{1}{N-1}\sum_{i<j}\|A_i - A_j\|.$$

Then the followings hold.

(1) $W \leq V^+ \leq U$

(2) $W + (N-2)U \geq (N-1)V^+$

(3) $\frac{N-1}{N-2}V^+ - \frac{1}{N-2}W \geq V^+ \geq \max\left\{\frac{N-1}{N-2}V^+ - \frac{1}{N-2}U, W\right\}$

(4) $\|\sum_{i=1}^N A_i\|^2 + (N-2)\sum_{i=1}^N \|A_i\|^2 = \sum_{i<j}\|A_i + A_j\|^2$

(5) $\sum_{i=1}^N \|A_i\|^2 \leq \frac{1}{N}\left(\sum_{i<j}\|A_i - A_j\|^2 + (V^+)^2\right)$

(6) $\sum_{i=1}^N \|A_i\|^2 \geq \frac{1}{N}\left\{(V^+)^2 + (V^-)^2\right\}$

Theorem 3.5 (Reverse Inequality of Sum Type Uncertainty Relation, [36]).

(1)

$$\begin{aligned} &\sum_{i=1}^N \sqrt{I_{A,B}^{(g,f)}(X_i)} \\ \leq\; & \frac{\sqrt{2}}{N-1}\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i \pm X_j)}\left\{\frac{\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A,B}^{(g,f)}(X_j)}}{\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A,B}^{(g,f)}(X_j)} \pm \mathrm{Re}\{\Gamma_{A,B}^{(g,f)}(X_i, X_j)\}}\right\}^{1/2}\end{aligned}$$

(2)

$$\begin{aligned} &\sum_{i=1}^N I_{A,B}^{(g,f)}(X_i) \\ \leq\; & \frac{2}{N-1}\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A.B}^{(g,f)}(X_j)}\left\{\frac{I_{A,B}^{(g,f)}(X_i \pm X_j)}{\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A,B}^{(g,f)}(X_j)} \pm \mathrm{Re}\{\Gamma_{A,B}^{(g,f)}(X_i, X_j)\}} - 1\right\}.\end{aligned}$$

Finally we give the sum type of uncertainty relation for entropy.

Theorem 3.6. *Let $P = (p_1, p_2, \ldots, p_N)$ and $Q = (q_1, q_2, \ldots, q_N)$ be two probability distributions. Then the entropies $H(P)$ and $H(Q)$ have the following uncertainty relation.*

$$H(P) + H(Q) \geq 2(1 - \max\{\sum_{i=1}^{N} p_i^2, \sum_{j=1}^{N} q_j^2\}).$$

Proof. By Corollary 2.4 in [37], we get

$$\sum_{i=1}^{N} \frac{2p_i(1-p_i)}{1+p_i} \leq H(P) \leq \sum_{i=1}^{N} \frac{1-p_i^2}{2}.$$

Then we have

$$\begin{aligned} H(P) + H(Q) &\geq \sum_{i=1}^{N} \frac{2p_i(1-p_i)}{1+p_i} + \sum_{j=1}^{N} \frac{2q_j(1-q_j)}{1+q_j} \\ &\geq \sum_{i=1}^{N} p_i(1-p_i) + \sum_{j=1}^{N} q_j(1-q_j) \\ &= 2 - (\sum_{i=1}^{N} p_i^2 + \sum_{j=1}^{N} q_j^2) \\ &\geq 2(1 - \max\{\sum_{i=1}^{N} p_i^2, \sum_{j=1}^{N} q_j^2\}). \end{aligned}$$

□

4. Generalized Quasi-Metric Adjusted Skew Information based Uncertainty Relations for Quantum Channels

For a quantum state $\rho \in M_{n,+,1}(\mathbb{C})$ and an arbitrary quantum channel Φ with Kraus representation $\Phi(\rho) = \sum_i K_i \rho K_i^*$, the coherence of quantum state ρ with respect ro the general quantum channel Φ is defined by

$$I(\rho, \Phi) = \sum_i I_{\rho,\rho}^{(g,f)}(K_i).$$

From the definition of $I(\rho, \Phi)$, we see that it depends on both the quantum state and the quantum channel, and characterizes some intrinsic feature of the state-channel interaction. Let

$\rho = \sum_{j=1}^{n} \lambda_j |\phi_j\rangle\langle\phi_j|$ be a spectral decomposition. Then

$$\begin{aligned} I(\rho,\Phi) &= \sum_i \sum_{j,,k} (m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j,\lambda_k)) |\langle\phi_j|K_i|\phi_k\rangle|^2 \\ &= \sum_i \sum_{j\neq k} (m_g(\lambda_j,\lambda_k) - m_{\Delta_g^f}(\lambda_j,\lambda_k)) |\langle\phi_j|K_i|\phi_k\rangle|^2. \end{aligned}$$

We state sum type uncertainty relation for general quantum channels.

Theorem 4.1. *Let Φ and Ψ be two quantum channels with Krause representaions $\Phi(\rho) = \sum_{i=1}^n E_i \rho E_i^*$, $\Psi(\rho) = \sum_{i=1}^n L_i \rho L_i^*$, respectively. Then*

$$\begin{aligned} & I(\rho,\Phi) + I(\rho,\Psi) \\ \geq\ & \max_{\pi\in S_n} \frac{1}{2} \sum_{i=1}^{n} \max\{I_{\rho,\rho}^{(g,f)}(E_i + L_{\pi(i)}), I_{\rho.\rho}^{(g,f)}(E_i - L_{\pi(i)})\}, \end{aligned}$$

where S_n is the n-element permutation group and $\pi \in S_n$ is an arbitrary n-element permutation.

The proof is given by Theorem 3.1 (1).

Let $g(x) = \frac{x+1}{2}$, $f(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha-1)(x^{1-\alpha}-1)}$ and $k = \frac{f(0)}{2} = \frac{\alpha(1-\alpha)}{2}$. Then we have

$$I(\rho,\Phi) = \frac{1}{2} \sum_i \sum_{j\neq k} (\lambda_j^\alpha - \lambda_k^\alpha)(\lambda_j^{1-\alpha} - \lambda_k^{1-\alpha}) |\langle\phi_j|K_i|\phi_k\rangle|^2.$$

We assume that $\alpha = \frac{1}{2}$ and give three examples.

Example 4.2. (1) Phase damping channel

$$\Phi(\rho) = \sum_{i=1}^{2} K_i \rho K_i^*$$

with

$$K_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|,\ K_2 = \sqrt{p}|1\rangle\langle 1|,\ 0 \leq p \leq 1.$$

(2) Amplitude damping channel

$$\Psi(\rho) = \sum_{i=1}^{2} L_i \rho L_i^*$$

with

$$L_1 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|,\ L_2 = \sqrt{p}|0\rangle\langle 1|,\ 0 \leq p \leq 1.$$

(3) Other channel

$$\Xi(\rho) = \sum_{i=1}^{2} E_i \rho E_i^*$$

with

$$E_1 = |0\rangle\langle 1| + \sqrt{1-p}|1\rangle\langle 0|,\ E_2 = \sqrt{p}|1\rangle\langle 0|,\ 0 \le p \le 1.$$

Then for an arbitrary qubit state $\rho = \frac{1}{2}(\mathbb{I} + r \cdot \sigma)$, where $\mathbb{I}$ is the identity operator, $r = (r_1, r_2, r_3)$ is a real three-dimensional vector such that $|r|^2 = r_1^2 + r_2^2 + r_3^3 \le 1$, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices, we have

$$I(\rho, \Phi) = \frac{(1-\sqrt{1-p})(r_1^2 + r_2^2)}{2s},$$

$$I(\rho, \Psi) = \frac{(1-\sqrt{1-p})(r_1^2 + r_2^2) + pr_3^2}{2s},$$

$$I(\rho, \Xi) = \frac{|r|^2 + r_3^2 - \sqrt{1-p}(r_1^2 - r_2^2)}{2s}$$

where $s = 1 + \sqrt{1 - |r|^2}$. These three quantities characterizes the difference of the three channels from an information-theoretic perspective.

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(K.Yanagi) Department of Mathematics, Josai University, 1-1 Keyakidai, Sakado 350-0295,Japan
Email address: yanagi@josai.ac.jp