Harmonic analysis with proofs

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Abstract

We survey some results related to trigonometric series of one variable, which are basic and classic in harmonic analysis. We give proofs for the results in detail.

1. Introduction

We review some basic, classic results in harmonic analysis. We focus on results related to trigonometric series. Proofs will be given for the results in detail.

In Section 2, we shall recall the definitions of the Dirichlet kernels $D_n(x)$, the conjugate Dirichlet kernels $\widetilde{D}_n(x)$, the Fejér kernels $K_n(x)$ and the conjugate Fejér kernels $\widetilde{K}_n(x)$ and we shall state some formulae including those kernels.

We shall consider some special trigomometric series with decreasing positive coefficients in Section 3. Among other things, we shall prove that the series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{\log(2+n)}$$

is a Fourier series, while

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\log(2+n)}$$

is not a Fourier series.

In Section 4, we shall prove the characterization due to Wiener of Fourier coefficients of functions of bounded variation with removable discontinuities.

In this note $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ stands for the set of positive integers.

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2. Some definitions and formulae for trigonometric functions

Definition 2.1. The Dirichlet kernels $D_n(x)$, $n \geq 0$, are defined as

$$D_n(x) = \frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx, \quad n \geq 1; \qquad D_0(x) = \frac{1}{2}.$$

Definition 2.2. We define the conjugate Dirichlet kernels $\widetilde{D}_n(x)$, $n \geq 0$, as

$$\widetilde{D}_n(x) = \sin x + \sin 2x + \cdots + \sin nx, \quad n \geq 1; \qquad \widetilde{D}_0(x) = 0.$$

Definition 2.3. The Fejér kernels $K_n(x)$, $n \geq 0$, are defined as

$$K_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} D_\nu(x).$$

Definition 2.4. We define the conjugate Fejér kernels $\widetilde{K}_n(x)$, $n \geq 0$, as

$$\widetilde{K}_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} \widetilde{D}_\nu(x).$$

Theorem 2.5. *Let $n \geq 1$. We have*

$$\frac{1}{2}a_0 + \sum_{m=1}^{n} a_m \cos mx = \sum_{\nu=0}^{n-1} D_\nu(x)\Delta a_\nu + a_n D_n(x),$$

where $\Delta a_\nu = a_\nu - a_{\nu+1}$ for $\nu \geq 0$.

Theorem 2.6. *Let $n \geq 2$. We see that*

$$\sum_{\nu=0}^{n-1} D_\nu(x)\Delta a_\nu = \sum_{m=0}^{n-2} (m+1)K_m(x)\Delta^2 a_m + nK_{n-1}(x)\Delta a_{n-1}, \tag{2.1}$$

where $\Delta^2 a_m = \Delta(\Delta a_m) = \Delta a_m - \Delta a_{m+1} = a_m - a_{m+1} - (a_{m+1} - a_{m+2}) = a_m - 2a_{m+1} + a_{m+2}$;

$$\frac{1}{2}a_0 + \sum_{m=1}^{n} a_m \cos mx = \sum_{m=0}^{n-2} (m+1)K_m(x)\Delta^2 a_m + nK_{n-1}(x)\Delta a_{n-1} + a_n D_n(x). \tag{2.2}$$

Theorem 2.7. *Let $n \geq 2$. We have*

$$\sum_{m=1}^{n} b_m \sin mx = \sum_{\nu=1}^{n-1} \widetilde{D}_\nu(x)\Delta b_\nu + b_n \widetilde{D}_n(x).$$

Theorem 2.8. *Let $n \geq 3$. We see that*

$$\sum_{\nu=1}^{n-1} \widetilde{D}_\nu(x)\Delta b_\nu = \sum_{m=1}^{n-2}(m+1)\widetilde{K}_m(x)\Delta^2 b_m + n\widetilde{K}_{n-1}(x)\Delta b_{n-1}; \tag{2.3}$$

$$\sum_{m=1}^{n} b_m \sin mx = \sum_{m=1}^{n-2}(m+1)\widetilde{K}_m(x)\Delta^2 b_m + n\widetilde{K}_{n-1}(x)\Delta b_{n-1} + b_n\widetilde{D}_n(x). \tag{2.4}$$

Proof of Theorem 2.5. Applying summation by parts arguments (see [3, Theorem 3.41, p.70]), we have

$$\begin{aligned}
\frac{1}{2}a_0 + \sum_{m=1}^{n} a_m \cos mx &= \frac{1}{2}a_0 + \sum_{m=1}^{n} a_m(D_m(x) - D_{m-1}(x)) \\
&= \frac{1}{2}a_0 + \sum_{m=1}^{n} a_m D_m(x) - \sum_{m=1}^{n} a_m D_{m-1}(x) \\
&= \sum_{\nu=0}^{n-1} a_\nu D_\nu(x) + a_n D_n(x) - \sum_{\nu=0}^{n-1} a_{\nu+1} D_\nu(x) \\
&= \sum_{\nu=0}^{n-1} D_\nu(x)\Delta a_\nu + a_n D_n(x).
\end{aligned}$$

□

Proof of Theorem 2.6. Proof of (2.1). We note that $D_\nu = (\nu+1)K_\nu - \nu K_{\nu-1}$ for $\nu \geq 1$, and $D_0 = K_0$. Thus

$$\begin{aligned}
&\sum_{\nu=0}^{n-1} D_\nu(x)\Delta a_\nu = K_0\Delta a_0 + \sum_{\nu=1}^{n-1}(\nu+1)K_\nu\Delta a_\nu - \sum_{\nu=1}^{n-1}\nu K_{\nu-1}\Delta a_\nu \\
&= K_0\Delta a_0 + \sum_{m=1}^{n-2}(m+1)K_m\Delta a_m + nK_{n-1}(x)\Delta a_{n-1} - \sum_{m=0}^{n-2}(m+1)K_m\Delta a_{m+1} \\
&= \sum_{m=0}^{n-2}(m+1)K_m(x)\Delta^2 a_m + nK_{n-1}(x)\Delta a_{n-1}.
\end{aligned}$$

This proves (2.1). The formula (2.2) follows from Theorem 2.5 and (2.1). □

Proof of Theorem 2.7. We note that $\sin mx = \widetilde{D}_m(x) - \widetilde{D}_{m-1}(x)$ for $m \geq 1$ and $\widetilde{D}_0(x) = 0$. Thus

$$\begin{aligned}
\sum_{m=1}^{n} b_m \sin mx &= \sum_{m=1}^{n} b_m(\widetilde{D}_m(x) - \widetilde{D}_{m-1}(x)) \\
&= \sum_{\nu=1}^{n} b_\nu\widetilde{D}_\nu(x) - \sum_{\nu=0}^{n-1} b_{\nu+1}\widetilde{D}_\nu(x) \\
&= \sum_{\nu=1}^{n-1}\widetilde{D}_\nu(x)\Delta b_\nu + b_n\widetilde{D}_n(x).
\end{aligned}$$

□

Proof of Theorem 2.8. Proof of (2.3). We note that $\widetilde{D}_\nu = (\nu+1)\widetilde{K}_\nu - \nu\widetilde{K}_{\nu-1}$ for $\nu \geq 1$, and $\widetilde{D}_0 = \widetilde{K}_0 = 0$. Thus

$$\begin{aligned}\sum_{\nu=1}^{n-1} \widetilde{D}_\nu(x)\Delta b_\nu &= \sum_{\nu=1}^{n-1}(\nu+1)\widetilde{K}_\nu \Delta b_\nu - \sum_{\nu=1}^{n-1} \nu\widetilde{K}_{\nu-1}\Delta b_\nu \\ &= \sum_{m=1}^{n-2}(m+1)\widetilde{K}_m\Delta b_m + n\widetilde{K}_{n-1}(x)\Delta b_{n-1} - \sum_{m=1}^{n-2}(m+1)\widetilde{K}_m\Delta b_{m+1} \\ &= \sum_{m=1}^{n-2}(m+1)\widetilde{K}_m(x)\Delta^2 b_m + n\widetilde{K}_{n-1}(x)\Delta b_{n-1}.\end{aligned}$$

This completes the proof of (2.3). The equation (2.4) follows from Theorem 2.7 and (2.3). □

For the Dirichlet kernels and the conjugate Dirichlet kernels, we have the following formulae.

Theorem 2.9 (Zygmund [6, p. 2]). *Let $n \geq 0$. We have*

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}.$$

Proof. The proof is needed only for the case $n \geq 1$. We express $2\sin\frac{1}{2}xD_n(x)$ by a telescoping series and see that

$$\begin{aligned}\sin\frac{1}{2}x + \sum_{\nu=1}^{n} 2\sin\frac{1}{2}x\cos\nu x &= \sin\frac{1}{2}x + \sum_{\nu=1}^{n}\left(\sin(\nu+\frac{1}{2})x - \sin(\nu-\frac{1}{2})x\right) \\ &= \sin(n+\frac{1}{2})x,\end{aligned}$$

which implies the conclusion. □

Theorem 2.10 (Zygmund [6, p. 2]). *Let $n \geq 0$. We have*

$$\widetilde{D}_n(x) = \frac{\cos\frac{1}{2}x - \cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}.$$

Proof. We may assume that $n \geq 1$. Similarly to the proof of Theorem 2.9, we have

$$\begin{aligned}\sum_{\nu=1}^{n} 2\sin\frac{1}{2}x\sin\nu x &= \sum_{\nu=1}^{n}\left(\cos(\nu-\frac{1}{2})x - \cos(\nu+\frac{1}{2})x\right) \\ &= \cos\frac{1}{2}x - \cos(n+\frac{1}{2})x,\end{aligned}$$

from which we deduce the conclusion. □

Applying Theorem 2.9, we have the following.

Theorem 2.11 (Zygmund [6, p. 88]). *For $n \geq 0$ we have*

$$\begin{aligned} K_n(t) &= \frac{1}{n+1}\sum_{\nu=0}^{n} D_\nu(t) = \frac{1}{n+1}\sum_{\nu=0}^{n} \frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t} \\ &= \frac{1}{n+1}\frac{1-\cos(n+1)t}{(2\sin\frac{1}{2}t)^2} \\ &= \frac{2}{n+1}\left\{\frac{\sin\frac{1}{2}(n+1)t}{2\sin\frac{1}{2}t}\right\}^2. \end{aligned}$$

Proof. The second equality follows from Theorem 2.9. We note that

$$\begin{aligned} \sum_{\nu=0}^{n} 2\sin\frac{1}{2}t\sin(\nu+\frac{1}{2})t &= \sum_{\nu=0}^{n}(\cos\nu t - \cos(\nu+1)t) \\ &= 1-\cos(n+1)t, \end{aligned}$$

which implies the third equality. The last equality follows by the formula $1-\cos\theta = 2\sin^2(\theta/2)$. □

Using Theorem 2.10, we have the following.

Theorem 2.12 (Zygmund [6, p. 91]). *Let $n \geq 0$. Then*

$$\begin{aligned} \widetilde{K}_n(t) &= \frac{1}{n+1}\sum_{\nu=0}^{n} \widetilde{D}_\nu(t) = \frac{1}{2}\cot\frac{1}{2}t - \frac{1}{n+1}\sum_{\nu=0}^{n}\frac{\cos(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t} \\ &= \frac{1}{2}\cot\frac{1}{2}t - \frac{1}{n+1}\frac{\sin(n+1)t}{(2\sin\frac{1}{2}t)^2}. \end{aligned}$$

Proof. The second equality follows from Theorem 2.10. We see that

$$\begin{aligned} \sum_{\nu=0}^{n} 2\sin\frac{1}{2}t\cos(\nu+\frac{1}{2})t &= \sum_{\nu=0}^{n}(\sin(\nu+1)t - \sin\nu t) \\ &= \sin(n+1)t, \end{aligned}$$

which implies the last equality. □

3. Special trigomometric series with decreasing positive coefficients

For results in this section we refer to [2, III].

We consider the series of the form

$$\sum_{n=1}^{\infty}\lambda_n e^{in\theta}, \tag{3.1}$$

where $\lambda_n > 0$, $\lambda_n \geq \lambda_{n+1}$ for all $n \geq 1$.

Theorem 3.1. *Let the series* $\sum \lambda_n e^{in\theta}$ *be as in* (3.1). *We further assume that* $\lambda_n \to 0$ *as* $n \to \infty$. *Then the series is uniformly convergent in any subset* I *of* $\mathbb{R}$ *such that* $\operatorname{dist}(I, 2\pi\mathbb{Z}) > 0$, *where* $2\pi\mathbb{Z} = \{2k\pi : k \in \mathbb{Z}\}$ *and* $\operatorname{dist}(E, F) = \inf_{x\in E, y\in F} |x - y|$ *for* $E, F \subset \mathbb{R}$.

Corollary 3.2. *Let* I *be a subset of* $\mathbb{R}$ *as in Theorem* 3.1. *Then each of the two series*

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos n\theta}{n}$$

converges uniformly on I.

This follows from Theorem 3.1 with $\lambda_n = 1/n$.

To prove Theorem 3.1 we need the following lemmas.

Lemma 3.3. *Let* $0 \le p \le q, p, q \in \mathbb{Z}$ *and* $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$. *Then*

$$\left| \sum_{n=p}^{q} e^{in\theta} \right| \le \frac{1}{|\sin \frac{1}{2}\theta|}.$$

Lemma 3.4. *Let* $0 \le p \le q, p, q \in \mathbb{Z}$ *and* $\lambda_n > 0, \lambda_n \ge \lambda_{n+1}$ *for* $n \ge 0, n \in \mathbb{Z}$. *Then*

$$\left| \sum_{n=p}^{q} \lambda_n e^{in\theta} \right| \le \frac{\lambda_p}{|\sin \frac{1}{2}\theta|},$$

for $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$.

Proof of Lemma 3.3. Summing up a geometric series, we have

$$\sum_{n=p}^{q} e^{in\theta} = e^{ip\theta} \sum_{n=0}^{q-p} e^{in\theta} = e^{ip\theta} \frac{1 - e^{i(q-p+1)\theta}}{1 - e^{i\theta}}.$$

We note that

$$|1 - e^{i\theta}|^2 = (1 - \cos\theta)^2 + \sin^2\theta = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}.$$

Thus

$$\left| \sum_{n=p}^{q} e^{in\theta} \right| \le \frac{2}{|1 - e^{i\theta}|} = \frac{1}{|\sin \frac{1}{2}\theta|}.$$

□

Proof of Lemma 3.4. Let

$$U_m = \sum_{n=p}^{m} e^{in\theta}.$$

Then by Lemma 3.3, $|U_m| \le 1/|\sin(\theta/2)|$. Applying summation by parts, we write

$$\sum_{n=p}^{q} \lambda_n e^{in\theta} = \lambda_p U_p + \lambda_{p+1}(U_{p+1} - U_p) + \cdots + \lambda_q(U_q - U_{q-1})$$
$$= U_p(\lambda_p - \lambda_{p+1}) + U_{p+1}(\lambda_{p+1} - \lambda_{p+2}) + \cdots + U_{q-1}(\lambda_{q-1} - \lambda_q) + \lambda_q U_q.$$

Thus, using $|U_m| \le 1/|\sin(\theta/2)|$ and $\lambda_m \ge \lambda_{m+1} \ge 0$, we see that

$$\left|\sum_{n=p}^{q} \lambda_n e^{in\theta}\right| \le \frac{1}{|\sin\frac{1}{2}\theta|}((\lambda_p - \lambda_{p+1}) + (\lambda_{p+1} - \lambda_{p+2}) + \cdots + (\lambda_{q-1} - \lambda_q) + \lambda_q)$$
$$= \frac{1}{|\sin\frac{1}{2}\theta|}\lambda_p.$$

□

Proof of Theorem 3.1. Let $\delta = \mathrm{dist}(I, 2\pi\mathbb{Z})$. Then, using the inequality $\sin\theta \ge (2/\pi)\theta$, $0 < \theta \le \pi/2$, we see that

(3.2) $$\sup_{\theta\in I} \frac{1}{|\sin\frac{1}{2}\theta|} \le \pi\delta^{-1}.$$

Given $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that $\lambda_{p_0}\pi\delta^{-1} < \epsilon$. Thus, if $p_0 \le p \le q$ and $\theta \in I$, by Lemma 3.4 and (3.2) we have

$$\left|\sum_{n=p}^{q} \lambda_n e^{in\theta}\right| \le \frac{1}{|\sin\frac{1}{2}\theta|}\lambda_p$$
$$\le \lambda_p\pi\delta^{-1} \le \lambda_{p_0}\pi\delta^{-1} < \epsilon.$$

Therefore the series $\sum \lambda_n e^{in\theta}$ is uniformly convergent on I by the Cauchy criterion. □

Theorem 3.5. *Suppose that $\lambda_n > 0$, $\lambda_n \ge \lambda_{n+1}$ for all $n \ge 1$ and $n\lambda_n \le C_0$ for all $n \ge 1$ with a constant C_0. Then the series $\sum_{n=1}^{\infty} \lambda_n \sin n\theta$ is boundedly convergent on $\mathbb{R}$.*

Corollary 3.6. *The series $\sum_{n=1}^{\infty} n^{-1} \sin n\theta$ is boundedly convergent on $\mathbb{R}$.*

Proof of Theorem 3.5. Let $U_N(\theta) = \sum_{n=1}^{N} \lambda_n \sin n\theta$. We show that $|U_N(\theta)| \le C$ for $\theta \in (0, \pi)$ with a constant C independent of θ and N. This implies that the same holds for $-\pi < \theta < 0$ by the oddness of the function U_N. Also, we have $U_N(-\pi) = U_N(0) = U_N(\pi) = 0$. It follows that $|U_N(\theta)| \le C$ for $\theta \in [-\pi, \pi]$. Thus the inequality is true for all $\theta \in \mathbb{R}$ by the 2π periodicity of U_N, which is what we need.

We split $\sum_{n=1}^{N} \lambda_n \sin n\theta$ into two pieces:

$$U_N(\theta) = \sum_{n=1}^{N} \lambda_n \sin n\theta = \sum_{n=1}^{M} \lambda_n \sin n\theta + \sum_{n=M+1}^{N} \lambda_n \sin n\theta = S_1 + S_2, \quad \text{say}.$$

By Lemma 3.4, we see that

$$(3.3)\qquad |S_2| \le \left|\sum_{n=M+1}^{N} \lambda_n e^{in\theta}\right| \le \frac{\lambda_{M+1}}{|\sin\frac{1}{2}\theta|} \le \frac{C_0}{M+1}\frac{1}{|\sin\frac{1}{2}\theta|}.$$

On the other hand, since $|\sin n\theta| \le n|\theta|$, we have

$$(3.4)\qquad |S_1| = \left|\sum_{n=1}^{M} \lambda_n \sin n\theta\right| \le \sum_{n=1}^{M} \lambda_n n\theta \le C_0 M\theta.$$

If we choose M so that $\theta^{-1} \le M < \theta^{-1}+1$, then by (3.4)

$$(3.5)\qquad |S_1| \le C_0(\theta^{-1}+1)\theta = C_0(1+\theta) \le C_0(1+\pi).$$

Also, since $\sin x \ge (2/\pi)x$ $(0 < x \le \pi/2)$, by (3.3)

$$(3.6)\qquad |S_2| \le C_0 \frac{1}{M+1}\frac{\pi}{\theta} \le C_0 \frac{1}{M+1} M\pi \le C_0\pi.$$

Combining (3.5) and (3.6), we have $|U_N(\theta)| \le C_0(1+2\pi)$, which completes the proof. □

Let $\lambda_n > 0$, $n = 0, 1, 2, \ldots$, $\lambda_n \ge \lambda_{n+1}$, $n \ge 0$, $\lambda_n \to 0$. We consider

$$(C)\quad f(\theta) = \frac{1}{2}\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \cos n\theta, \qquad (S)\quad g(\theta) = \sum_{n=1}^{\infty} \lambda_n \sin n\theta.$$

By Theorem 3.1, each of the two series is uniformly convergent on any compact subset of $[-\pi,\pi] \setminus \{0\}$ and so f, g are continuous on $[-\pi,\pi] \setminus \{0\}$.

Theorem 3.7. *Suppose that $f \in L^1([-\pi,\pi])$. Then the series in (C) is the Fourier series of f. Also, if $g \in L^1([-\pi,\pi])$, then the series in (S) is the Fourier series of g.*

Proof. Suppose that $g \in L^1([-\pi,\pi])$ and $m \in \mathbb{N}$. We note that the series $\sum_{n=1}^{\infty} \lambda_n \sin n\theta \sin m\theta$ converges uniformly on $[-\pi,\pi]$. This can be seen through the Cauchy criterion by using Lemma 3.4 as follows:

$$\left|\sum_{n=p}^{q} \lambda_n \sin n\theta \sin m\theta\right| \le \frac{\lambda_p}{|\sin\frac{1}{2}\theta|}|\sin m\theta| \le \lambda_p \frac{m|\theta|}{(2/\pi)|(1/2)\theta|} = \lambda_p m\pi,$$

where the second inequality follows from the inequalities $|\sin x| \le |x|$ and $|\sin y| \ge (2/\pi)|y|$ $(|y| \le \pi/2)$. Thus we can apply term by term integration and get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta) \sin m\theta \, d\theta = \sum_{n=1}^{\infty} \lambda_n \frac{1}{\pi}\int_{-\pi}^{\pi} \sin n\theta \sin m\theta \, d\theta = \lambda_m$$

for $m \in \mathbb{N}$. This is what we have claimed for (S).

Next, let $f \in L^1([-\pi,\pi])$. As in the case of (S), the series

$$\frac{1}{2}\lambda_0(1-\cos m\theta) + \sum_{n=1}^{\infty} \lambda_n \cos n\theta (1-\cos m\theta)$$

converges uniformly on $[-\pi,\pi]$, since

$$\begin{aligned}\left|\sum_{n=p}^{q} \lambda_n \cos n\theta(1-\cos m\theta)\right| &\leq \frac{\lambda_p}{|\sin\frac{1}{2}\theta|}|(1-\cos m\theta)| \\ &\leq \lambda_p \frac{(1/2)(m\theta)^2}{(2/\pi)|(1/2)\theta|} = \lambda_p m^2 \pi |\theta|/2,\end{aligned}$$

for $m \geq 1$, where for the second inequality the estimate $|1-\cos x| \leq (1/2)x^2$ is also used. Thus, integrating term by term, for $m \geq 1$ we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)(1-\cos m\theta)\, d\theta = \lambda_0 - \sum_{n=1}^{\infty} \lambda_n \frac{1}{\pi}\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \lambda_0 - \lambda_m. \tag{3.7}$$

Letting $m \to \infty$ and using the Riemann-Lebesgue lemma (see [4, p.103]) and our assumption that $\lambda_m \to 0$, we have $\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = \lambda_0$. Using this in (3.7), we have $\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos m\theta\, d\theta = \lambda_m$ for all $m \geq 1$. This completes the proof. □

Theorem 3.8. *Let $\{\lambda_n\}$ be as in the definition of the series (C) and (S). Set*

$$\Lambda = \sum_{n=1}^{\infty} \frac{\lambda_n}{n}.$$

(1) *If $\Lambda < \infty$, then $f, g \in L^1([-\pi,\pi])$ and the series in (C) and (S) are Fourier series of f and g, respectively.*
(2) *If $g \in L^1([-\pi,\pi])$ and the series in (S) is the Fourier series of g, then $\Lambda < \infty$.*

Proof. Proof of part (1). Let $\Lambda_k = \sum_{n=1}^{k} \lambda_n$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\Lambda_k}{k(k+1)} &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{n=1}^{k} \lambda_n \\ &= \sum_{n=1}^{\infty} \lambda_n \sum_{k=n}^{\infty} \frac{1}{k(k+1)} \\ &= \Lambda.\end{aligned}$$

Let f and g be as in (C) and (S) and define $h = f + ig$. Let k satisfy $\pi/(k+1) \leq \theta < \pi/k$, for $\theta \in (0,\pi)$. We write

$$h = \frac{1}{2}\lambda_0 + \sum_{n=1}^{\infty} \lambda_n e^{in\theta} = \frac{1}{2}\lambda_0 + \sum_{n=1}^{k-1} \lambda_n e^{in\theta} + \sum_{n=k}^{\infty} \lambda_n e^{in\theta}$$

and by Lemma 3.4 we see that

$$|h| \le \frac{1}{2}\lambda_0 + \sum_{n=1}^{k-1} \lambda_n + \frac{\lambda_k}{\sin \frac{1}{2}\theta} \le \frac{1}{2}\lambda_0 + \Lambda_k + \frac{\pi\lambda_k}{\theta} \le \frac{1}{2}\lambda_0 + \Lambda_k + (k+1)\lambda_k.$$

Thus

$$\begin{aligned}\int_0^{\pi} |h(\theta)|\, d\theta &= \sum_{k=1}^{\infty} \int_{\pi/(k+1)}^{\pi/k} |h(\theta)|\, d\theta \\ &\le \sum_{k=1}^{\infty} (\frac{1}{2}\lambda_0 + \Lambda_k) \frac{\pi}{k(k+1)} + \sum_{k=1}^{\infty} \frac{\pi\lambda_k}{k} \\ &= \pi(\frac{1}{2}\lambda_0 + 2\Lambda).\end{aligned}$$

Thus, if $\Lambda < \infty$, then $f, g \in L^1([-\pi, \pi])$. Therefore, Theorem 3.7 implies part (1).

Proof of part (2). Suppose that $g \in L^1([-\pi, \pi])$ and $\lambda_m = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta) \sin m\theta \, d\theta$. Then

$$\sum_{m=1}^{N} \frac{\lambda_m}{m} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sum_{m=1}^{N} \frac{\sin m\theta}{m}\, d\theta. \tag{3.8}$$

Since $\sum_{m=1}^{\infty} \frac{\sin m\theta}{m}$ converges boundedly by Theorem 3.5, letting $N \to \infty$ in (3.8), we have

$$\sum_{m=1}^{\infty} \frac{\lambda_m}{m} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sum_{m=1}^{\infty} \frac{\sin m\theta}{m}\, d\theta < \infty.$$

□

Here we recall some results on numerical series.

Lemma 3.9. *Let $\{v_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Let $\Delta v_n = v_n - v_{n+1}$ and $\Delta^2 v_n = \Delta(\Delta v_n)$, $n \ge 0$.*

(1) *If $v_n \to 0$, then $\sum_{n=0}^{\infty} \Delta v_n = v_0$.*

(2) *If $nv_n \to 0$ and either $\sum_{n=0}^{\infty} v_n$ or $\sum_{n=0}^{\infty}(n+1)\Delta v_n$ is convergent, then $\sum_{n=0}^{\infty} v_n = \sum_{n=0}^{\infty}(n+1)\Delta v_n$.*

(3) *If $v_n \ge v_{n+1}$, $v_n \ge 0$ for $n \ge 0$ and $\sum_{n=0}^{\infty} v_n < \infty$, then $nv_n \to 0$.*

(4) *If $v_n \to 0$ and $\{v_n\}$ is convex, which means that $\Delta^2 v_n \ge 0$ for $n \ge 0$, then $\Delta v_n \ge 0$, $n \ge 0$, $n\Delta v_n \to 0$ and*

$$\sum_{n=0}^{\infty}(n+1)\Delta^2 v_n = \sum_{n=0}^{\infty} \Delta v_n = v_0. \tag{3.9}$$

Proof. Proof of part (1). We see that $\sum_{n=0}^{N} \Delta v_n = v_0 - v_{N+1}$. So letting $N \to \infty$, we get the conclusion.

Proof of part (2). We note that

$$\sum_{n=0}^{N}(n+1)\Delta v_n = \sum_{n=0}^{N}(n+1)v_n - \sum_{n=1}^{N+1} n v_n = \sum_{n=0}^{N} v_n - (N+1)v_{N+1},$$

which implies the claim.

Proof of part (3). We have

$$\sum_{k=[n/2]}^{n} v_k \geq (n-[n/2]+1)v_n \geq ((n/2)+1)v_n \geq (n/2)v_n \geq 0.$$

It follows that $nv_n \to 0$.

Proof of part (4). Since $\{\Delta v_n\}$ is decreasing and converges to 0, we have $\Delta v_n \geq 0$. Since $v_n \to 0$, by part (1) $\sum \Delta v_n$ is convergent. Thus by part (3) we see that $n\Delta v_n \to 0$. Therefore we can apply part (2) and have the first equality of (3.9); the second equality follows from part (1). □

Theorem 3.10. *Suppose that $\{\lambda_n\}$ is convex. Then f in (C) is non-negative and integrable; further the series in (C) is the Fourier series of f.*

Proof. By (2.2) of Theorem 2.6, Theorem 2.9 and Theorem 2.11 we have for $0 < \theta < \pi$

$$\begin{aligned}
&\frac{1}{2}\lambda_0 + \sum_{m=1}^{n} \lambda_m \cos m\theta \\
&= \sum_{m=0}^{n-2}(m+1)K_m(\theta)\Delta^2\lambda_m + nK_{n-1}(\theta)\Delta\lambda_{n-1} + \lambda_n D_n(\theta) \\
&= \frac{1}{4\sin^2\frac{1}{2}\theta}\left(\sum_{\nu=0}^{n-2}(1-\cos(\nu+1)\theta)\Delta^2\lambda_\nu + (1-\cos n\theta)\Delta\lambda_{n-1}\right) + \lambda_n \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.
\end{aligned}$$

Since $\lambda_n \to 0$ and $\Delta\lambda_{n-1} \to 0$ as $n \to \infty$, letting $n \to \infty$, we have

$$f(\theta) = \frac{1}{4\sin^2\frac{1}{2}\theta}\sum_{\nu=0}^{\infty}(1-\cos(\nu+1)\theta)\Delta^2\lambda_\nu.$$

Obviously, $f(\theta) \geq 0$ and by $\int_0^\pi K_\nu(\theta)\,d\theta = \pi/2$, applying Lemma 3.9 (4), we have

$$\int_0^\pi f(\theta)\,d\theta = (\pi/2)\sum_{\nu=0}^{\infty}(\nu+1)\Delta^2\lambda_\nu = (\pi/2)\lambda_0.$$

Thus by Theorem 3.7 the series in (C) is the Fourier series of f. □

Corollary 3.11. *We have the following results.*

(1) *The series*

$$\sum_{n=1}^{\infty} \frac{\cos nx}{\log(2+n)}$$

is a Fourier series.

(2) *The series*

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\log(2+n)}$$

is not a Fourier series.

Proof. Since the sequence $\{(\log(2+n))^{-1}\}$ is convex, by Theorem 3.10 we have part (1).

To prove part (2), let $g(x) = \sum_{n=1}^{\infty} \sin nx/\log(2+n)$ (the series is convergent pointwise). We recall the fact that $\sum_{n=1}^{\infty} 1/(n\log(2+n)) = \infty$. If there exists $h \in L^1([-\pi,\pi])$ such that the series $\sum_{n=1}^{\infty} \sin nx/\log(2+n)$ is the Fourier series of h, then it is known that $h = g$. Thus $g \in L^1([-\pi,\pi])$, which would imply by Theorem 3.8 (2) that $\sum_{n=1}^{\infty} 1/(n\log(2+n)) < \infty$. Thus we reach a contradiction. This completes the proof of part (2). □

By Corollary 3.11 we can see that the conjugate function (the Hilbert transform) of an integrable function need not be integrable. To confirm this it may be helpful to consult [2, Theorem 76] where a relation between the existence of the Hilbert transform and Abel summability of the conjugate series is investigated.

The series in Corollary 3.2 can be expressed as follows.

Theorem 3.12. *Let* $0 < \theta < 2\pi$. *Then*

$$\cos\theta + \frac{1}{2}\cos(2\theta) + \frac{1}{3}\cos(3\theta) + \ldots = -\log\left(2\sin\frac{\theta}{2}\right), \tag{3.10}$$

$$\sin\theta + \frac{1}{2}\sin(2\theta) + \frac{1}{3}\sin(3\theta) + \ldots = \frac{\pi-\theta}{2}. \tag{3.11}$$

Definition 3.13. Let $w \in \mathbb{C} \setminus (-\infty, 0]$. Then we have a unique $\theta \in (-\pi,\pi)$ such that

$$\frac{w}{|w|} = e^{i\theta}.$$

We define $\operatorname{Arg} w = \theta$.

Definition 3.14. For $x > 0$, $\ln x$ is defined as

$$\ln x = \int_1^x \frac{1}{y}\,dy.$$

Definition 3.15. For $w \in \mathbb{C} \setminus (-\infty, 0]$, $\log w$ is defined as

$$\log w = \ln |w| + i \operatorname{Arg} w.$$

We note that $\log x = \ln x$ for $x > 0$.

To prove Theorem 3.12 we need the following two lemmas.

Lemma 3.16. *Let $z \in \mathbb{C}$, $|z| \le 1$, $z \ne 1$. Then*

$$-\log(1-z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \dots,$$

where log *is as in Definition* 3.15 (*we note that* $1 - z \in \{w \in \mathbb{C} : |w - \frac{1}{2}| \le \frac{1}{2}\} \setminus \{0\}$).

Lemma 3.17. *Let $T : (-\pi/2, \pi/2) \to \mathbb{R}$ be the bijection defined by $T = \tan|(-\pi/2, \pi/2)$ (the restriction of* tan *to $(-\pi/2, \pi/2)$). Suppose that $z = x + iy$, $x = \operatorname{Re} z > 0$. Then*

$$\operatorname{Arg} z = T^{-1}\left(\frac{y}{x}\right),$$

where $T^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ is the inverse mapping of T.

Proof of Lemma 3.16. We use the equation

$$\frac{1}{1-x} - (1 + x + x^2 + \dots + x^n) = \frac{x^{n+1}}{1-x}, \quad 0 \le x < 1.$$

Integration of both sides gives

$$\int_0^x \frac{1}{1-y}\,dy - (x + \frac{1}{2}x^2 + \dots + \frac{1}{n+1}x^{n+1}) = \int_0^x \frac{y^{n+1}}{1-y}\,dy$$

for $0 \le x < 1$. By changing variables and Definition 3.14, we see that

$$\int_0^x \frac{1}{1-y}\,dy = -\int_1^{1-x} \frac{1}{y}\,dy = -\ln(1-x) = -\log(1-x).$$

Thus

$$\left| -\log(1-x) - (x + \frac{1}{2}x^2 + \dots + \frac{1}{n+1}x^{n+1}) \right| \le \frac{1}{1-x}\int_0^1 y^{n+1}\,dy = \frac{1}{1-x}\frac{1}{n+2}.$$

Letting $n \to \infty$, we see that

$$(3.12) \qquad -\log(1-x) = x + \frac{1}{2}x^2 + \dots + \frac{1}{n}x^n + \dots, \quad 0 \le x < 1.$$

Let

$$F(z) = z + \frac{1}{2}z^2 + \dots + \frac{1}{n}z^n + \dots$$

for $|z| < 1$. Then F is holomorphic in $|z| < 1$ and it is known that $-\log(1-z)$ is also holomorphic in $|z| < 1$. Thus by the uniqueness of analytic continuation and (3.12), we have $-\log(1-z) = F(z)$ for $|z| < 1$.

The series defining $F(z)$ is also convergent if $|z| = 1$ and $z \neq 1$ (see Corollary 3.2 and also [3, Theorem 3.44, Chap. 3]). Thus by Abel's theorem (see [3, Theorem 8.2, Chap. 8]), if $z = e^{i\theta}$, $0 < \theta < 2\pi$, we can define $F(e^{i\theta})$ by continuity as

$$F(e^{i\theta}) = \lim_{r \to 1, r<1} F(re^{i\theta}) = e^{i\theta} + \frac{1}{2}e^{2i\theta} + \cdots + \frac{1}{n}e^{ni\theta} + \cdots.$$

Since $-\log(1-z) = F(z)$ for $|z| < 1$ and $-\log(1-z)$ is continuous at $z = e^{i\theta}$, $0 < \theta < 2\pi$, we have $F(e^{i\theta}) = -\log(1 - e^{i\theta})$ for $0 < \theta < 2\pi$. This completes the proof of Lemma 3.16. □

Proof of Lemma 3.17. We have $-\pi/2 < \operatorname{Arg} z < \pi/2$, since $\operatorname{Re} z > 0$. If $\operatorname{Arg} z = \theta$, by Definition 3.13 we have $|z|^{-1}z = e^{i\theta} = \cos\theta + i\sin\theta$, which can be rewritten as

$$\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}} = \cos\theta + i\sin\theta.$$

It follows that $\tan\theta = \sin\theta/\cos\theta = y/x$. Since $-\pi/2 < \theta < \pi/2$, we have $\tan\theta = T(\theta)$. Thus $T(\theta) = y/x$ and hence $\operatorname{Arg} z = \theta = T^{-1}(y/x)$. □

Proof of Theorem 3.12. Let $z = e^{i\theta}$ with $0 < \theta < 2\pi$. Then by Lemma 3.16 we see that

$$-\left(\log|1 - e^{i\theta}| + i\operatorname{Arg}(1 - e^{i\theta})\right) = \sum_{n=1}^{\infty} \frac{e^{ni\theta}}{n} = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} + i\sum_{n=1}^{\infty} \frac{\sin n\theta}{n}.$$

Comparing real and imaginary parts, we have

$$-\log|1 - e^{i\theta}| = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n}, \tag{3.13}$$

$$-\operatorname{Arg}(1 - e^{i\theta}) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}. \tag{3.14}$$

We note that

$$|1 - e^{i\theta}|^2 = (1 - \cos\theta)^2 + \sin^2\theta = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}.$$

Using this in (3.13), we have (3.10).

Next, since $\operatorname{Re}(1 - e^{i\theta}) > 0$ and $1 - e^{i\theta} = 1 - \cos\theta - i\sin\theta$, by Lemma 3.17 we have $\operatorname{Arg}(1 - e^{i\theta}) = T^{-1}(-\sin\theta/(1 - \cos\theta))$. We note that

$$-\frac{\sin\theta}{1-\cos\theta} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\frac{\sin(\frac{\pi}{2} - \frac{\theta}{2})}{\cos(\frac{\pi}{2} - \frac{\theta}{2})} = \tan\left(\frac{\theta - \pi}{2}\right).$$

Since $0 < \theta < 2\pi$, we have $-\pi/2 < (\theta - \pi)/2 < \pi/2$. Thus $\tan(\theta - \pi)/2 = T((\theta - \pi)/2)$. Therefore

$$T^{-1}\left(-\frac{\sin\theta}{1-\cos\theta}\right) = T^{-1}\left(\tan\frac{\theta-\pi}{2}\right) = T^{-1} \circ T\left(\frac{\theta-\pi}{2}\right) = \frac{\theta-\pi}{2}.$$

Thus we have $\operatorname{Arg}(1 - e^{i\theta}) = (\theta - \pi)/2$. Applying this in (3.14), we have (3.11). This completes the proof of Theorem 3.12. □

For the continuity of the function g in (S) we have the following result.

Theorem 3.18. *Let $g(\theta) = \sum_{n=1}^{\infty} \lambda_n \sin(n\theta)$ be as in (S); we recall that the series is convergent for every $\theta \in \mathbb{R}$. Then the following three statements are equivalent.*

(1) *The function g is continuous on $[0, 2\pi]$.*

(2) *The series $\sum_{n=1}^{\infty} \lambda_n \sin(n\theta)$ is uniformly convergent on $[0, 2\pi]$.*

(3) $\lim_{n\to\infty} n\lambda_n = 0$.

Lemma 3.19. *We consider $g(\theta) = \sum_{n=1}^{\infty} \lambda_n \sin(n\theta)$ as in (S). Suppose that $g \in L^1([0, 2\pi])$. Then*

$$\int_0^\theta g(t)\,dt = \sum_{n=1}^{\infty} \int_0^\theta \lambda_n \sin(nt)\,dt = \sum_{n=1}^{\infty} \frac{\lambda_n(1-\cos(n\theta))}{n}.$$

Proof. We note that

$$\begin{aligned}\int_0^{2\pi} g(x+\theta)\sin nx\,dx &= \int_0^{2\pi} g(x)\sin n(x-\theta)\,dx \\ &= \cos n\theta \int_0^{2\pi} g(x)\sin nx\,dx - \sin n\theta \int_0^{2\pi} g(x)\cos nx\,dx \\ &= \pi\lambda_n \cos n\theta,\end{aligned}$$

where the last equality follows from Theorem 3.7. Let $G(x) = \int_0^x g(t)\,dt$. Then G is 2π periodic, which can be seen from $G(x+2\pi) - G(x) = \int_x^{x+2\pi} g(t)\,dt = \int_0^{2\pi} g(t)\,dt = 0$. Since the series $\sum_{n=1}^{\infty} n^{-1}\sin nx$ is boundedly convergent to $(\pi - x)/2$ by Corollary 3.6 and (3.11), we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n}\int_0^{2\pi} g(x+\theta)\sin nx\,dx &= \int_0^{2\pi} g(x+\theta)\sum_{n=1}^{\infty}\frac{\sin nx}{n}\,dx \\ &= \int_0^{2\pi} g(x+\theta)\frac{\pi - x}{2}\,dx \\ &= \left[G(x+\theta)\frac{\pi - x}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} G(x+\theta)\,dx \\ &= -\pi G(\theta) + \frac{1}{2}\int_0^{2\pi} G(x)\,dx,\end{aligned}$$

where the penultimate equality follows by integration by parts and

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi} G(x)\,dx &= \left[G(x)\frac{x-\pi}{2}\right]_0^{2\pi} - \int_0^{2\pi} g(x)\frac{x-\pi}{2}\,dx \\ &= \int_0^{2\pi} g(x)\frac{\pi - x}{2}\,dx \\ &= \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{2\pi} g(x)\sin nx\,dx = \sum_{n=1}^{\infty}\frac{1}{n}\pi\lambda_n.\end{aligned}$$

Thus

$$G(\theta)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda_n-\sum_{n=1}^{\infty}\frac{1}{n}\lambda_n\cos n\theta,$$

which implies the conclusion. □

Proof of Theorem 3.18. We first prove that (3) implies (2) then we prove (1) implies (3). This will conclude the proof of the theorem since it is well known that (2) implies (1).

Suppose that we have (3). Then for any $\epsilon>0$ there exists a positive integer N such that $n\lambda_n<\epsilon$ if $n\ge N$. For $p\ge N$ and $q>p$ by Lemma 3.4 we have

$$\left|\sum_{n=p}^{q}\lambda_n\sin n\theta\right|\le\frac{\lambda_p}{|\sin\frac{1}{2}\theta|}\le\lambda_p\frac{\pi}{\theta}$$

for $0<\theta\le\pi$. Letting $q\to\infty$, if $\theta\in[1/p,\pi]$, we see that

$$\left|\sum_{n=p}^{\infty}\lambda_n\sin n\theta\right|\le\pi p\lambda_p\le\pi\epsilon. \tag{3.15}$$

If $\theta<1/p$ and $0<\theta\le\pi$, let $p\le q\le 1/\theta<q+1$. Then using (3.15) we have

$$\begin{aligned}\left|\sum_{n=p}^{\infty}\lambda_n\sin n\theta\right|&\le\left|\sum_{n=p}^{q}\lambda_n\sin n\theta\right|+\left|\sum_{n=q+1}^{\infty}\lambda_n\sin n\theta\right|\\&\le\sum_{n=p}^{q}\lambda_n n\theta+\epsilon\pi\le\epsilon\theta q+\epsilon\pi\le(1+\pi)\epsilon.\end{aligned}$$

Since $\sum_{n=p}^{\infty}\lambda_n\sin n\theta=0$ for $\theta=0$, we have for $\theta\in[0,1/p)$

$$\left|\sum_{n=p}^{\infty}\lambda_n\sin n\theta\right|\le(1+\pi)\epsilon \tag{3.16}$$

if $p\ge N$. By (3.15) and (3.16) we see that for $\theta\in[0,\pi]$

$$\left|\sum_{n=p}^{\infty}\lambda_n\sin n\theta\right|\le(1+\pi)\epsilon$$

whenever $p\ge N$, which implies that the series $\sum_{n=1}^{\infty}\lambda_n\sin n\theta$ is uniformly convergent on $[0,\pi]$. By this we see that $\sum_{n=1}^{\infty}\lambda_n\sin n\theta$ is uniformly convergent on $[-\pi,\pi]$ since $\sum_{n=1}^{\infty}\lambda_n\sin n\theta$ is odd, which implies (2) since $\sum_{n=1}^{\infty}\lambda_n\sin n\theta$ is 2π periodic.

We now prove that (1) implies (3). Suppose that g is continuous on $[0,\pi]$. Then $g(\theta)\to g(0)=0$ as $\theta\to 0$. By Lemma 3.19 we have

$$\lim_{\theta\to 0}\frac{1}{\theta}\int_0^{\theta}g(t)\,dt=\lim_{\theta\to 0}\sum_{n=1}^{\infty}\frac{\lambda_n(1-\cos(n\theta))}{n\theta}.$$

Taking $\theta = \pi/(2k)$ and using the inequality: $1 - \cos x \geq (2/\pi^2)x^2$, $0 \leq x \leq \pi$, we see that

$$\begin{aligned} 0 = \lim_{k\to\infty} \sum_{n=k}^{2k} \frac{\lambda_n(1-\cos(n\pi/(2k)))}{n\pi/(2k)} &\geq \lim_{k\to\infty} \frac{2k}{\pi}\lambda_{2k}\frac{2}{\pi^2}\sum_{n=k}^{2k} n\frac{\pi^2}{(2k)^2} \\ &\geq \lim_{k\to\infty} \frac{2k}{\pi}\lambda_{2k}\frac{2}{\pi^2}\frac{\pi^2}{(2k)^2}k^2 \\ &= \lim_{k\to\infty}\frac{k}{\pi}\lambda_{2k}. \end{aligned}$$

It follows that $\lim_{k\to\infty} 2k\lambda_{2k} = 0$, which also implies that $\lim_{k\to\infty}(2k+1)\lambda_{2k+1} = 0$ on account of the monotonicity of λ_k. Altogether, we have $\lim_{k\to\infty} k\lambda_k = 0$. □

4. Characterization of Fourier coefficients for continuous functions of bounded variation

A variant of the following result will be used in proving Theorem 4.6 below.

Theorem 4.1. *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of non-negative real numbers such that $A_k \leq 1/k$ for all $k \in \mathbb{N}$. Then the following three conditions are equivalent:*

(1)

$$\lim_{n\to\infty} n\sum_{k=1}^{\infty} A_k^2 \sin^2\left(\frac{k\pi}{2n}\right) = 0,$$

(2)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} k^2A_k^2 = 0,$$

(3)

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} kA_k = 0.$$

Proof. Part (2) implies part (3). We assume part (2). Then By the Schwarz inequality, we have

$$\frac{1}{n}\sum_{k=1}^{n} kA_k \leq \left(\frac{1}{n}\sum_{k=1}^{n} k^2A_k^2\right)^{1/2} \to 0 \quad \text{as } n\to\infty.$$

This implies part (3). Here we do not use the condition $A_k \leq 1/k$.

Part (2) follows from part (3). Applying the condition $A_k \leq 1/k$ and using part (3), we see that

$$\frac{1}{n}\sum_{k=1}^{n} k^2A_k^2 \leq \frac{1}{n}\sum_{k=1}^{n} kA_k \to 0 \quad \text{as } n\to\infty,$$

which is part (2).

Part (1) implies part (2). Using the inequality $\sin x \geq (2/\pi)x$, $0 \leq x \leq \pi/2$, we see that

$$n\sum_{k=1}^{\infty} A_k^2 \sin^2\left(\frac{k\pi}{2n}\right) \geq n\sum_{k=1}^{n} A_k^2 \sin^2\left(\frac{k\pi}{2n}\right) \geq n\sum_{k=1}^{n} A_k^2 \left(\frac{2}{\pi}\frac{k\pi}{2n}\right)^2$$
$$= \frac{1}{n}\sum_{k=1}^{n} k^2 A_k^2,$$

from which we see that part (1) implies part (2). Here we do not use the condition $A_k \leq 1/k$.

Part (2) implies part (1). We write $k = 2nm + \ell$, $0 \leq \ell < 2n$. Then

$$\sin^2\left(\frac{(2nm+\ell)\pi}{2n}\right) = \sin^2\left(m\pi + \frac{\ell}{2n}\pi\right) = \sin^2\left(\frac{\ell}{2n}\pi\right).$$

Using this and the inequality, $\sin x \leq x$, $x \geq 0$, we have, letting $A_0 = 0$,

$$\begin{aligned} n\sum_{k=1}^{\infty} A_k^2 \sin^2\left(\frac{k\pi}{2n}\right) &= n\sum_{m=0}^{\infty}\sum_{\ell=0}^{2n-1} A_{2mn+\ell}^2 \sin^2\left(\frac{\ell\pi}{2n}\right) \\ &\leq (\pi/2)^2 n \sum_{m=0}^{\infty}\sum_{\ell=0}^{2n-1} A_{2mn+\ell}^2 \left(\frac{\ell}{n}\right)^2 \\ &\leq (\pi/2)^2 \frac{1}{n}\sum_{m=0}^{\infty}\sum_{\ell=0}^{2n-1} \ell^2 A_{2mn+\ell}^2 \\ &\leq (\pi/2)^2 \sum_{m=0}^{\infty} p_{m,n}, \end{aligned}$$

where

$$p_{m,n} = \frac{1}{n}\sum_{\ell=0}^{2n-1} \ell^2 A_{2mn+\ell}^2.$$

We see that

$$\begin{aligned} p_{m,n} &\leq \frac{1}{n}\sum_{\ell=0}^{2n-1} (2mn+\ell)^2 A_{2mn+\ell}^2 \\ &\leq \frac{1}{n}\sum_{k=2mn}^{2(m+1)n} k^2 A_k^2 \\ &\leq \frac{1}{n}\sum_{k=1}^{2(m+1)n} k^2 A_k^2 \\ &\leq \frac{2n(m+1)}{n}\frac{1}{2n(m+1)}\sum_{k=1}^{2(m+1)n} k^2 A_k^2 \quad \to 0 \quad (n\to\infty), \end{aligned}$$

where the convergence to 0 in the last line follows if we assume part (2). Also, using the inequality $A_k \leq 1/k$, we have

$$p_{m,n} \leq \frac{1}{n} \sum_{\ell=1}^{2n-1} \ell^2 (2mn+\ell)^{-2} \leq C(m+1)^{-2}$$

for $m \geq 0$. Here C is a constant independent of n. Thus by the dominated convergence theorem of Lebesgue we have

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} p_{m,n} = 0.$$

This implies part (1) under the condition in part (2). □

Definition 4.2. Let $f : [0, 2\pi] \to \mathbb{C}$. Let $P = \{x_j\}_{j=0}^m$ with $0 = x_0 < x_1 < \cdots < x_m = 2\pi$ be a partition of the interval $[0, 2\pi]$. We say that f is a function of bounded variation if

$$\|f\|_{BV([0,2\pi])} = \sup_P \sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| < \infty,$$

where the supremum is taken over all partitions P of $[0, 2\pi]$. (See [1, p.97].)

For a integrable function f and $k \in \mathbb{Z}$, let

$$C_k(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt}\, dt \tag{4.1}$$

be the Fourier coefficient.

Theorem 4.3. *If f is a 2π periodic function on $\mathbb{R}$ which is of bounded variation on $[0, 2\pi]$, then for all $k \in \mathbb{Z} \setminus \{0\}$ we have*

$$|C_k(f)| \leq \frac{1}{4|k|} \|f\|_{BV([0,2\pi])}.$$

Proof. Let k be a positive integer. We easily see that

$$2|C_k(f)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \left(f(t+(j-1)\pi/k) - f(t+j\pi/k)\right) e^{-ikt}\, dt \right|, \quad 1 \leq j \leq 2k.$$

Thus, summing over j, $1 \leq j \leq 2k$, we have

$$4k|C_k(f)| \leq \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^{2k} |f(t+(j-1)\pi/k) - f(t+j\pi/k)|\, dt \leq \|f\|_{BV([0,2\pi])},$$

which implies that $|C_k(f)| \leq \frac{1}{4k} \|f\|_{BV([0,2\pi])}$. The result for the case $k < 0$ follows from this if we observe that $C_{-k}(f) = C_k(\widetilde{f})$ and $\|\widetilde{f}\|_{BV([0,2\pi])} = \|f\|_{BV([0,2\pi])}$ with $\widetilde{f}(x) = f(-x)$. □

It is known that there exists a continuous function f of bounded variation with period 2π for which we do not have $|kC_k(f)| \to 0$ ($|k| \to \infty$). Cantor's function can be used to construct an example (see [2, p.27]). On the other hand we have the following results (Theorems 4.4, 4.6) of Wiener (see [2, pp. 27-28] and also [5, p. 153, p. 196]).

Theorem 4.4. *Let f be a 2π-periodic function on $\mathbb{R}$. Suppose that f is of bounded variation on $[0, 2\pi]$. Then we have the following.*

(1) *If f is continuous on $\mathbb{R}$, then*

$$\lim_{n\to\infty} n \sum_{k=-\infty}^{\infty} |C_k(f)|^2 \sin^2\left(\frac{k\pi}{2n}\right) = 0, \tag{4.2}$$

where $C_k(f)$ is as in (4.1).

(2) *If (4.2) holds, then $f(x+0) = f(x-0)$ for all $x \in \mathbb{R}$; and hence the discontinuities of f should be of the first kind and removable.*

Let

$$F_n(x) = \sum_{m=1}^{2n} \left| f\left(x + \frac{m}{n}\pi\right) - f\left(x + \frac{m-1}{n}\pi\right)\right|^2. \tag{4.3}$$

To prove Theorem 4.4 we need the following.

Lemma 4.5. *Let $C_k(f)$ be as in (4.1) and F_n as in (4.3). Then*

$$\int_0^{2\pi} |F_n(x)|\, dx = 16\pi n \sum_{k=-\infty}^{\infty} |C_k(f)|^2 \sin^2\left(\frac{k\pi}{2n}\right).$$

Proof. We have

$$\begin{aligned}
\int_0^{2\pi} F_n(x)\, dx &= \sum_{m=1}^{2n} \int_0^{2\pi} \left| f\left(x + \frac{m}{n}\pi\right) - f\left(x + \frac{m-1}{n}\pi\right)\right|^2 dx \\
&= 2n \int_0^{2\pi} \left| f\left(x + \frac{1}{2n}\pi\right) - f\left(x - \frac{1}{2n}\pi\right)\right|^2 dx \\
&= 4\pi n \sum_{k=-\infty}^{\infty} |C_{n,k}|^2,
\end{aligned}$$

where the last equality follows by the Parseval theorem (see [3, Theorem 11.40, p. 328]) with

$$\begin{aligned}
C_{n,k} &= \frac{1}{2\pi} \int_0^{2\pi} \left(f\left(x + \frac{\pi}{2n}\right) - f\left(x - \frac{\pi}{2n}\right)\right) e^{-ikx}\, dx \\
&= C_k(f) \left(e^{ik\pi/(2n)} - e^{-ik\pi/(2n)}\right) \\
&= C_k(f) 2i \sin(k\pi/(2n)).
\end{aligned}$$

Collecting the results, we get the conclusion of the lemma. □

Proof of Theorem 4.4. Let

$$\omega_f(\tau) = \sup_{x\in\mathbb{R}, |y|\le\tau} |f(x+y) - f(x)|.$$

Then $F_n(x) \le \omega_f(\pi/n)\|f\|_{BV([0,2\pi])}$. Thus, if f is continuous, $F_n(x) \to 0$ uniformly, so $\int_0^{2\pi} F_n(x)\,dx \to 0$ as $n \to \infty$. This implies part (1) by Lemma 4.5.

Proof of part (2). Since f is 2π-periodic, we may assume that $x \in [0, 2\pi]$ in the conclusion. Suppose that there exists $x_0 \in [0, 2\pi]$ such that

$$|f(x_0 + 0) - f(x_0 - 0)| > d \quad \text{for some } d > 0.$$

Then there exists $\delta > 0$ such that if $|y - x_0| < \delta$, $|z - x_0| < \delta$ and $y < x_0 < z$, then $|f(z) - f(y)| > d/2$. Suppose that $2\pi/n < \delta$. For $x \in [0, 2\pi]$ we have either

(i) $x_0 \in [x, x + 2\pi]$

or

(ii) $x_0 + 2\pi \in [x, x + 2\pi]$.

First we consider the case (i). We deal with the following three cases separately:

(a) $x_0 \in (x + (m_0 - 1)\pi/n, x + (m_0 + 1)\pi/n)$ for some $1 \le m_0 \le 2n - 1$;

(b) $x_0 \in (x - \pi/n, x + \pi/n)$;

(c) $x_0 \in (x + (2n - 1)\pi/n, x + (2n + 1)\pi/n)$.

We now handle the case (a). We have $|f(x+(m_0+1)\pi/n) - f(x+(m_0-1)\pi/n)| > d/2$ and hence

$$|f(x + m_0\pi/n) - f(x + (m_0 - 1)\pi/n)| > d/4$$

or

$$|f(x + (m_0 + 1)\pi/n) - f(x + m_0\pi/n)| > d/4.$$

Next, we treat the case (b). Then $|f(x + \pi/n) - f(x - \pi/n)| > d/2$, which implies that we have

$$|f(x) - f(x - \pi/n)| > d/4$$

or

$$|f(x + \pi/n) - f(x)| > d/4.$$

Since f is 2π-periodic, it follows that

$$|f(x + (2n\pi)/n) - f(x + (2n - 1)\pi/n)| > d/4$$

or

$$|f(x + \pi/n) - f(x)| > d/4.$$

Finally, in the case (c) we have $|f(x+(2n+1)\pi/n) - f(x+(2n-1)\pi/n)| > d/2$ and so

$$|f(x+(2n+1)\pi/n) - f(x+(2n\pi)/n)| > d/4$$

or

$$|f(x+(2n\pi)/n) - f(x+(2n-1)\pi/n)| > d/4,$$

which implies that

$$|f(x+\pi/n) - f(x)| > d/4$$

or

$$|f(x+(2n\pi)/n) - f(x+(2n-1)\pi/n)| > d/4.$$

Collecting results in the cases (a), (b) and (c), we see that $F_n(x) \geq (d/4)^2$ for all $x \in [0, 2\pi]$ and $n > 2\pi/\delta$ in the case (i).

The same holds also in the case (ii). This can be seen as follows. Let $y_0 = x_0 + 2\pi$. Then $y_0 \in [x, x+2\pi]$ and

$$|f(y_0+0) - f(y_0-0)| = |f(x_0+0) - f(x_0-0)| > d.$$

Thus we can apply the arguments in the case (i) with y_0 in place of x_0.

Combining results, we have $F_n(x) \geq (d/4)^2$ for all $x \in [0, 2\pi]$ and $n > 2\pi/\delta$. Therefore, if $\|F_n\|_1 \to 0$ as $n \to \infty$, which holds if we have (4.2) by Lemma 4.5, then we have $f(x+0) = f(x-0)$ for all x. This completes the proof of part (2). □

By Theorems 4.3, 4.4 and by applying Theorem 4.1 suitably, we have the following.

Theorem 4.6. *Let f be a 2π-periodic function on $\mathbb{R}$ which is of bounded variation on $[0, 2\pi]$. Then we have the following.*

(1) *If f is continuous on $\mathbb{R}$, then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=-n}^{n} |kC_k(f)| = 0, \tag{4.4}$$

where $C_k(f)$ is as in (4.1).

(2) *Suppose that (4.4) holds. Then $f(x+0) = f(x-0)$ for all $x \in \mathbb{R}$.*

We note that (4.4) holds if $\lim_{|k|\to\infty} |kC_k(f)| = 0$.

Remark 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π periodic and such that

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \in [-\pi, \pi] \setminus 0. \end{cases}$$

Then f is of bounded variation and $C_k(f) = 0$ for all $k \in \mathbb{Z}$ and so we have (4.4), but f is not continuous at $x = 0$.

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