Weak and strong convergence theorems for families of nonexpansive mappings in Banach spaces

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Abstract

In this paper, we study the asymptotic behavior of orbits of nonexpansive semigroups in Banach spaces. We also prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

1. Introduction

Let E be a real Banach space, let C be a nonempty subset of E. For a mapping $T : C \to E$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The existence of fixed points of nonexpansive mappings in Banach and metric spaces has been investigated since the early 1960s (For example, see [8, 9, 10, 13, 19]). The behavior of the sequence of Picard iterates of T is one of the important problems in metric fixed point theory because this allows us to approximate a fixed point in the simplest way. Moreau [23] proved that if C is a closed subset of a Hilbert space and if $F(T)$ has nonempty interior, then for each $x \in C$, the sequence $\{T^n x\}$ converges strongly to a point in $F(T)$. Kirk and Sims [18] generalized this result to Banach spaces which are strictly convex and the nonempty closed subsets of which are densely proximal. Grzesik, Kaczor, Kuczumow and Reich [15] proved convergence of iterates of nonexpansive mappings: Let C be a bounded closed and convex subset of a uniformly convex Banach space E. Assume that C has nonempty interior and that it is locally uniformly rotund. Let T be a nonexpansive mapping of C into itself and let $x \in C$. If T has no fixed point in the interior of C, then there exists a unique point z_0 on the boundary ∂C of C such that each sequence $\{T^n x : n = 1, 2, 3, \dots\}$ converges strongly to z_0. They [15] also proved the convergence of orbits of one-parameter nonexpansive semigroups.

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Baillon [6] proved the following first nonlinear mean convergence theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$,

$$\{S_n x\} = \left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^i x \right\}$$

converges weakly to a fixed point of T (see also [31]).

In recent years, a new direction has been very active essentially after the publication of Ran and Reurings results [25]. They proved an analogue of the classical Banach contraction principle [7] in metric spaces endowed with a partial order. In particular, they show how this extension is useful when dealing with some special matrix equations (see also [17, 24, 32, 33].) Bin Dehaish and Khamsi [14] proved a weak convergence theorem of Mann's type [22] for monotone nonexpansive mappings in Banach spaces endowed with a partial order (see also [22, 26]). Shukla and Wiśnicki [29] obtained a nonlinear mean convergence theorem for monotone nonexpansive mappings in such Banach spaces.

In this paper, we study the asymptotic behavior of orbits of nonexpansive semigroups with no common fixed points in the interior of their domains. We also prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings in uniformly convex Banach spaces endowed with a partial order.

2. Preliminaries and Lemmas

Throughout this paper, we assume that E is a real Banach space with norm $\|\cdot\|$. We denote by E^* the topological dual space of E. We denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{R}^+$ the set of all nonnegative real numbers. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in E converges weakly to x. We also denote by $\langle y, x^* \rangle$ the value of $x^* \in E^*$ at $y \in E$. For a subset A of E, co$A$ and $\overline{\text{co}}A$ mean the convex hull of A and the closure of convex hull of A, respectively.

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real-valued functions defined on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. For each $s \in S$ and $g \in B(S)$, we can define an element $\ell_s g \in B(S)$ by $(\ell_s g)(t) = g(st)$ for all $t \in S$. We also denote by ℓ_s^* the conjugate operator of ℓ_s. Let $C(S)^*$ be the dual space of $C(S)$. A linear functional μ on $C(S)$ is called a mean on $C(S)$ if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(g(t))$ or $\int g(t)d\mu(t)$ instead of $\mu(g)$ for $\mu \in C(S)^*$ and $g \in C(S)$. A mean μ on $C(S)$ is called invariant if $\mu(\ell_s g) = \mu(g)$ for all $s \in S$ and $g \in C(S)$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(g) = g(s)$ for every $g \in B(S)$. A convex combination of point evaluations is called a finite mean on S. A finite mean μ on S is also a mean on $C(S)$ containing constants.

The following definition which was introduced by Takahashi [30] is crucial in the nonlinear ergodic theory for abstract semigroups (see also [16]). Let h be a continuous function of S into E such that the weak closure of $\{h(t) : t \in S\}$ is weakly compact. Then, for any $\mu \in C(S)^*$ there exists a unique element $h_\mu \in E$ such that

$$\langle h_\mu, x^* \rangle = (\mu)_t \langle h(t), x^* \rangle = \int \langle h(t), x^* \rangle \, d\mu(t)$$

for all $x^* \in E^*$. If μ is a mean on $C(S)$, then h_μ is contained in $\overline{\text{co}}\{h(t) : t \in S\}$ (for example, see [30, 31]). Sometimes, h_μ will be denoted by $\int h(t)d\mu(t)$.

Throughout this paper, we assume that S is a commutative semitopological semigroup with identity. Let C be a closed convex subset of a Banach space E. Then, a family $\mathcal{S} = \{T(s) : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(a) $T(s+t) = T(s)T(t)$ for all $s, t \in S$;
(b) $s \mapsto T(s)x$ is continuous;
(c) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $T(t), t \in S$. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Assume that for each $x \in C$ and $x^* \in E^*$, the weak closure of $\{T(t)x : t \in S\}$ is weakly compact. Let μ be a mean on $C(S)$. Following [27], we also write $T_\mu x$ instead of $\int T(t)x \, d\mu(t)$ for $x \in C$. We remark that T_μ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathcal{S})$. If μ is a finite mean, i.e.,

$$\mu = \sum_{i=1}^{n} a_i \delta_{t_i} \ (t_i \in S, a_i \geq 0, \ \sum_{i=1}^{n} a_i = 1),$$

then

$$T_\mu x = \sum_{i=1}^{n} a_i T(t_i)x.$$

A Banach space E is said to be strictly convex if $\dfrac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space, we have that if

$$\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$$

for $x, y \in E$ and $\lambda \in (0,1)$, then $x = y$. For every ε with $0 \leq \varepsilon \leq 2$, we define the modulus $\delta(\varepsilon)$ of convexity of E by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. If E is uniformly convex, then for r, ε with $r \geq \varepsilon > 0$, we have $\delta\left(\frac{\varepsilon}{r}\right) > 0$ and

$$\left\|\frac{x+y}{2}\right\| \leq r\left(1 - \delta\left(\frac{\varepsilon}{r}\right)\right)$$

for every $x, y \in E$ with $\|x\| \leq r,\ \|y\| \leq r$ and $\|x-y\| \geq \varepsilon$. It is well-known that a uniformly convex Banach space is reflexive and strictly convex. Let $S_E = \{x \in E : \|x\| = 1\}$ be a unit sphere in a Banach space E. A Banach space E is said to be *locally uniformly rotund* if for each $x \in S_E$ and for each $\varepsilon \in (0,2]$, there exists $\delta(x,\varepsilon) > 0$ such that for each $y \in S_E$ with $\|x-y\| \geq \varepsilon$, we have

$$1 - \left\|\frac{x+y}{2}\right\| \geq \delta(x,\varepsilon).$$

For more details, see [20].

Let E be a Banach space, let C be a nonempty bounded closed and convex subset of E. Assume that C has nonempty interior, that is, $\operatorname{int}(C) \neq \emptyset$. We say that C is *locally uniformly rotund* if for each $x \in \partial C$ and for each $\varepsilon \in (0, d_x)$, where $d_x = \sup\{\|x - x'\| : x' \in C\}$, there exists $\delta(x,\varepsilon) > 0$ such that for each $y \in C$ with $\|x-y\| \geq \varepsilon$, we have

$$\operatorname{dist}\left(\frac{x+y}{2}, \partial C\right) := \inf\left\{\left\|\frac{x+y}{2} - x'\right\| : x' \in \partial C\right\} \geq \delta(x,\varepsilon).$$

Let C be a nonempty bounded closed and convex subset of a Banach space E. Assume that C has nonempty interior, that is, $\operatorname{int}(C) \neq \emptyset$. We say that C is uniformly convex if for each $\varepsilon \in (0, \operatorname{diam(C)})$, there exists $\eta_C(\varepsilon) > 0$ such that for each $x, y \in C$ with $\|x-y\| \geq \varepsilon$, we have

$$\operatorname{dist}\left(\frac{x+y}{2}, \partial C\right) := \inf\left\{\left\|\frac{x+y}{2} - x'\right\| : x' \in \partial C\right\} \geq \eta_C(\varepsilon).$$

Observe that if a Banach space E admits a nonempty bounded closed and convex subset which has nonempty interior and is uniformly convex, then E has to be reflexive (see [31]).

Now, we present a simple example of a bounded closed and convex subset of a Hilbert space, which is locally uniformly rotund but not uniformly convex (see [20]).

Example 2.1 ([20])**.** Let $H = \ell^2$. Let

$$C = \left\{ x = \{x^i\} \in H = \ell^2 : \sum_{k=2}^{\infty} \left(|x^{2k-1}|^k + |x^{2k}|^k\right)^{\frac{2}{k}} \leq 1 \right\}.$$

Then, C is bounded, closed, convex and has nonempty interior. Moreover, C is locally uniformly rotund, but not uniformly convex.

Let C be a nonempty subset of a Banach space E and let T be mapping of C into E. The mapping T is said to be *demiclosed* if for any sequence $\{x_n\} \subset C$ the following implication hold:

$$\text{w-}\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} \|Tx_n - y\| = 0$$

imply that

$$Tx = y$$

(see [12]).

Theorem 2.2 ([12])**.** *Let C be a nonempty closed and convex subset of a uniformly convex Banach space E. Let T be a nonexpansive mapping of C into itself and let I be the identity mapping. Then, $I - T$ is demiclosed at 0, that is,*

$$\text{w-}\lim_{n\to\infty} x_n = x \textit{ and } \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

imply that

$$Tx = x.$$

The following theorem has been essentially established in [13] (see also [8, 10, 19, 31]).

Theorem 2.3 ([13])**.** *Let C be a nonempty bounded closed and convex subset of a uniformly convex Banach space E. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Then, $F(\mathcal{S})$ is nonempty.*

The following theorem has been essentially established in [9] (see also [10, 13, 31]).

Theorem 2.4 ([9])**.** *Let C be a closed and convex subset of a strictly convex Banach space E. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Then, the set $F(\mathcal{S})$ is closed and convex.*

The following lemma plays an important role in this paper (see [5, 16, 28]).

Lemma 2.5 ([5])**.** *Let C be a nonempty bounded, closed convex subset of a uniformly convex Banach space E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T(t) :$*

$t \in S\}$ be a nonexpansive semigroup on C. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$ for each $s \in S$. Then, for each $t \in S$,

$$\lim_{n\to\infty} \sup_{y\in C} \|T_{\mu_n} y - T(t) T_{\mu_n} y\| = 0.$$

3. Asymptotic behavior of orbits of nonexpansive semigroups

In this section, we prove convergence theorems for nonexpansive semigroups with no common fixed points in the interior of their domains. Throughout this paper, we assume that S is a commutative semitopological semigroup with identity.

3.1. **Convergence theorems for nonexpansive semigroups.** In this subsection, we prove strong convergence theorems for nonexpansive semigroups. A sequence $\{x_n\}$ in C is said to be an *approximating sequence* of a nonexpansive mapping T of C into itself if

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

(for example, see [15, 21, 31]). A sequence $\{x_n\}$ in C is said to be an *approximating sequence* of a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ on C if

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$$

for each $t \in S$ (for example, see [15]). We study the behavior of approximating sequences of nonexpansive semigroups.

Theorem 3.1 ([1]). *Let E be a reflexive Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T(t){:}t \in S\}$ be a nonexpansive semigroup on C. Assume that $I - T(t)$ is demiclosed at 0 for each $t \in S$. If $\mathcal{S} = \{T(t){:}t \in S\}$ has a unique common fixed point z_0 and z_0 lies on the boundary ∂C of C, then every approximating sequence $\{x_n\}$ of $\mathcal{S}$ converges strongly to z_0.*

We can prove convergence of orbits of nonexpansive semigroups with no common fixed points in the interior of their domains.

Theorem 3.2 ([1]). *Let E be a uniformly convex Banach space and let C be a bounded closed and convex subset of E. Assume that C has nonempty interior and that it is locally uniformly rotund. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. If $\mathcal{S} = \{T(t) : t \in S\}$ has no common fixed point in the* *interior of C, then there exists a unique point z_0 on the boundary ∂C of C such that each orbit $\{T(t)x : t \in S\}$ converges strongly to z_0.*

The following example shows that the assumption that C is locally uniformly rotund is crucial (see[15]).

Example 3.3. Let $H = \mathbb{R}^2$ be endowed with the standard Euclidean norm and let $C = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$. If $T(x, y) = (1, -y)$ for $(x, y) \in C$, then T is nonexpansive and $(1, 0) \in \partial C$ is its unique fixed point, but $\{T^n(1, 1), n = 1, 2, \ldots\}$, do not converge to $(1, 0)$.

3.2. **Deduced theorems from main results.** Using Theorems 3.1 and 3.2, we get some convergence theorems (see [15]).

Let C be a closed convex subset of a Banach space E. Then, a family $\mathcal{S} = \{T(s) : s \in \mathbb{R}^+\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(a) $T(s + t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;
(b) $T(0)x = x$ for each $x \in C$;
(c) $s \mapsto T(s)x$ is continuous;
(d) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \in \mathbb{R}^+$

Using Theorem 3.1 and Lemma 2.5, we obtain the following convergence theorem (see also [11]).

Theorem 3.4 ([1]). *Let E be a uniformly convex Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Assume that $\mathcal{S} = \{T(t) : t \in S\}$ has a unique common fixed point z_0 and that z_0 lies on the boundary ∂C of C. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$$

for each $s \in S$. Let $x \in C$ and let $\{z_n\}$ be the sequence defined by

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) T_{\mu_n} z_n \quad \text{for each} \quad n \in \mathbb{N}.$$

Then, $\{z_n\}$ converges strongly to z_0.

Using Theorem 3.1 and Lemma 2.5, we also obtain the following convergence theorem (see also [34]).

Theorem 3.5 ([1]). *Let E be a uniformly convex Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund.*

Let $\mathcal{S} = \{T(t){:}t \in S\}$ be a nonexpansive semigroup on C. Assume that $\mathcal{S} = \{T(t){:}t \in S\}$ has a unique common fixed point z_0 and that z_0 lies on the boundary ∂C of C. Let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0$$

for each $s \in S$. Let $u_0 = x \in C$ and let $\{u_n\}$ be the sequence defined by

$$u_n = \frac{1}{n} u_{n-1} + \left(1 - \frac{1}{n}\right) T_{\mu_n} u_n \quad \textit{for each} \quad n \in \mathbb{N}.$$

Then, $\{u_n\}$ converges strongly to z_0.

Using Theorem 3.1, we obtain the following convergence theorems (see [15]).

Theorem 3.6. *Let E be a reflexive Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let T be a nonexpansive mapping of C into itself. Assume that $I - T$ is demiclosed at 0. If T has a unique fixed point z_0 and z_0 lies on the boundary ∂C of C, then every approximating sequence $\{x_n\}$ of T converges strongly to z_0.*

Theorem 3.7. *Let E be a reflexive Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let $\mathcal{S} = \{T(t){:}t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C. Assume that $I - T(t)$ is demiclosed at 0 for each $t \in \mathbb{R}^+$. If $\mathcal{S} = \{T(t){:}t \in \mathbb{R}^+\}$ has a unique common fixed point z_0 and z_0 lies on the boundary ∂C of C, then every approximating sequence $\{x_n\}$ of $\mathcal{S}$ converges strongly to z_0.*

Theorem 3.8. *Let E be a reflexive Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let T be a nonexpansive mapping of C into itself. Assume that $I - T$ is demiclosed at 0 and that T has a unique fixed point z_0 which lies on the boundary ∂C of C. Let $x \in C$ and let $\{z_n\}$ be the sequence defined by*

$$z_n = \frac{1}{n} x + \left(1 - \frac{1}{n}\right) T z_n \quad \textit{for each} \quad n \in \mathbb{N}.$$

Then, $\{z_n\}$ converges strongly to z_0.

Theorem 3.9. *Let E be a reflexive Banach space and let C be a bounded, closed and convex subset of E with nonempty interior. Assume further that C is locally uniformly rotund. Let T be a nonexpansive mapping of C into itself. Assume that $I - T$ is demiclosed at 0 and that T has a* *unique fixed point* z_0 *which lies on the boundary* ∂C *of* C*. Let* $u_0 = x \in C$ *and let* $\{u_n\}$ *be the sequence defined by*

$$u_n = \frac{1}{n}u_{n-1} + \left(1 - \frac{1}{n}\right)Tu_n \quad \textit{for each} \quad n \in \mathbb{N}.$$

Then, $\{u_n\}$ *converges strongly to* z_0*.*

By Theorem 3.4, we get the following convergence theorem (see also [31]).

Theorem 3.10. *Let* E *be a uniformly convex Banach space and let* C *be a bounded, closed and convex subset of* E *with nonempty interior. Assume further that* C *is locally uniformly rotund. Let* $\mathcal{S} = \{T(t): t \in \mathbb{R}^+\}$ *be a one-parameter nonexpansive semigroup on* C*. Assume that* $\mathcal{S} = \{T(t): t \in \mathbb{R}^+\}$ *has a unique common fixed point* z_0 *and that* z_0 *lies on the boundary* ∂C *of* C*. Let* $\{t_n\}$ *be a sequence in* $(0, \infty)$ *with* $t_n \to \infty$*. Let* $x \in C$ *and let* $\{z_n\}$ *be the sequence defined by*

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt \quad \textit{for each} \quad n \in \mathbb{N}.$$

Then, $\{z_n\}$ *converges strongly to* z_0*.*

By Theorem 3.5, we get the following theorem.

Theorem 3.11. *Let* E *be a uniformly convex Banach space and let* C *be a bounded, closed and convex subset of* E *with nonempty interior. Assume further that* C *is locally uniformly rotund. Let* $\mathcal{S} = \{T(t): t \in \mathbb{R}^+\}$ *be a one-parameter nonexpansive semigroup on* C*. Assume that* $\mathcal{S} = \{T(t): t \in \mathbb{R}^+\}$ *has a unique common fixed point* z_0 *and that* z_0 *lies on the boundary* ∂C *of* C*. Let* $\{t_n\}$ *be a sequence in* $(0, \infty)$ *with* $t_n \to \infty$*. Let* $u_0 = x \in C$ *and let* $\{u_n\}$ *be the sequence defined by*

$$u_n = \frac{1}{n}u_{n-1} + \left(1 - \frac{1}{n}\right)\frac{1}{t_n}\int_0^{t_n} T(t)u_n dt \quad \textit{for each} \quad n \in \mathbb{N}.$$

Then, $\{u_n\}$ *converges strongly to* z_0*.*

Using Theorem 3.2, we also get the following theorems (see [15]).

Theorem 3.12. *Let* E *be a uniformly convex Banach space and let* C *be a bounded closed and convex subset of* E*. Assume that* C *has nonempty interior and that it is locally uniformly rotund. Let* T *be a nonexpansive mapping of* C *into itself. If* T *has no fixed point in the interior of* C*, then there exists a unique point* z_0 *on the boundary* ∂C *of* C *such that each sequence* $\{T^n x : n = 1, 2, 3, \dots\}$ *converges strongly to* z_0*.*

Theorem 3.13. *Let* E *be a uniformly convex Banach space and let* C *be a bounded closed and convex subset of* E*. Assume that* C *has nonempty interior and that it is locally uniformly rotund.*

Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C. If $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ has no common fixed point in the interior of C, then there exists a unique point z_0 on the boundary ∂C of C such that each orbit $\{T(t)x : t \in \mathbb{R}^+\}$ converges strongly to z_0.

4. Nonlinear mean convergence theorems

In this section, we show nonlinear mean convergence theorems for two monotone nonexpansive mappings.

4.1. **Monotone and approximating sequences.** Throughout this section, we assume that E is a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E, that is,

$$x \preceq y \text{ implies } x + z \preceq y + z,$$

$$x \preceq y \text{ implies } \lambda x \preceq \lambda y$$

for every $x, y, z \in E$ and $\lambda \geq 0$. As usual we adopt the convention $x \succeq y$ if and only if $y \preceq x$. It follows that all *order intervals* $[x, \rightarrow] = \{z \in E : x \preceq z\}$ and $[\leftarrow, y] = \{z \in E : z \in E : z \preceq y\}$ are convex. Moreover, we assume that each order intervals $[x, \rightarrow]$ and $[\leftarrow, y]$ are closed. Recall that an order interval is any of the subsets

$$[a, \rightarrow] = \{x \in E; a \preceq x\} \quad \text{or} \quad [\leftarrow, a] = \{x \in E; x \preceq a\}.$$

for any $a \in E$. As a direct consequence of this, the subset

$$[a, b] = \{x \in E; a \preceq x \preceq b\} = [a, \rightarrow] \cap [\leftarrow, b]$$

is also closed and convex for each $a, b \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and endowed with a *partial order* $\preceq$ compatible with the linear structure of E. Let C be a nonempty subset of E. A mapping $T : C \rightarrow C$ is called *monotone* if

$$Tx \preceq Ty$$

for each $x, y \in C$ such that $x \preceq y$. For a mapping $T : C \rightarrow C$, we denote by $F(T)$ the set of *fixed points* of T, i.e., $F(T) = \{z \in C : Tz = z\}$.

In this subsection, we study monotone sequences and approximating sequences of nonexpansive mappings. A sequence $\{x_n\}$ in E is said to be *monotone* if

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots$$

(see also [14]). The following lemma was obtained by the author and Takahashi [3] (see also [4]).

Lemma 4.1 ([3])**.** *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself with $ST = TS$. Then,*

$$\lim_{n\to\infty} \sup_{x\in C} \left\| \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x - T\left(\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x \right) \right\| = 0$$

and

$$\lim_{n\to\infty} \sup_{x\in C} \left\| \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x - S\left(\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{i=0}^{n-1} S^i T^j x \right) \right\| = 0.$$

The following theorem was proved by Browder [12].

Theorem 4.2 ([12])**.** *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E and let T be a nonexpansive mapping of C into itself. Let $\{x_n\}$ be a sequence in C such that it converges weakly to an element u in C and $\{x_n - Tx_n\}$ converges strongly to 0. Then, u is a fixed point of T.*

Using Theorem 4.2, we can prove the following result which is crucial in this paper.

Theorem 4.3. *Let C be a nonempty bounded closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself. Let $\{x_n\}$ be a sequence in C which is a monotone, and approximating sequence of T and S, i.e.,*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Then, then the sequence $\{x_n\}$ converges weakly to a point of $F(S) \cap F(T)$.

4.2. **Nonlinear mean convergence theorems for nonexpansive mappings.** In this subsection, we show nonlinear mean convergence theorems for monotone nonexpansive mappings. Using Lemma 4.1, we can prove the following lemma which plays an important role in our results.

Lemma 4.4 ([2])**.** *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Let $x \in C$. For each $n \in \mathbb{N}$ and $m \in \{0, 1, 2, \ldots\}$, let*

$$U_n^{(m)} x = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^{k+m} T^{l+m} x.$$

Then, the sequence $\{U_n^{(m)} x\}_{n=1}^{\infty}$ in C is an approximating sequence of S and T uniformly in $m \in \{0, 1, 2, \ldots\}$.

Lemma 4.5 ([2]). *Let C be a nonempty closed convex subset of an ordered Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Let $x \in C$. For each $m \in \{0, 1, 2, \ldots\}$, let*

$$U_n^{(m)} x = \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^{k+m} T^{l+m} x.$$

Then, for each $m \in \{0, 1, 2, \ldots\}$, the sequence $\{U_n^{(m)} x\}_{n=1}^{\infty}$ in C is monotone.

We can prove a nonlinear mean convergence theorem for two monotone nonexpansive mappings.

Theorem 4.6 ([2]). *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E. Let S and T be monotone nonexpansive mappings of C into itself such that $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$. Assume that $x \preceq Sx$ and $x \preceq Tx$ for each $x \in C$. Then,*

$$\left\{ \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x \right\}$$

converges weakly to a point of $F(S) \cap F(T)$.

Using Theorem 4.6, we get some convergence theorems for monotone nonexpansive mappings in ordered uniformly convex Banach spaces (see [29]).

Theorem 4.7. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{S_n x\} = \{\frac{1}{n} \sum_{k=0}^{n-1} T^k x\}$ converges weakly to a point of $F(T)$.*

Theorem 4.8. *Let C be a nonempty closed convex subset of an ordered uniformly convex Banach space E and let T be a monotone nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Assume that $x \preceq Tx$ for each $x \in C$. Then, $\{T^n x\}$ converges weakly a point of $F(T)$.*

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