Shape invariance of K-stability of C^*-algebras

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Abstract

The present note records an observation that the K-stability of C^*-algebras is invariant under shape equivalence in the sense of Blackadar [1].

1 Introduction

For a unital C^*-algebra A, $\mathrm{GL}_n(A)$ denotes the group of all invertible $n \times n$-matrices over A. It is, as a topological space, homotopy equivalent to the group of unitaries $U_n(A)$. The canonical inclusion

$$\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A) \tag{1}$$

is defined by $a \mapsto \mathrm{diag}(a, 1_A),\ a \in \mathrm{GL}_n(A)$. A unital C^*-algebra A is said to be K-*stable* ([7]) if for each $n \geq 1$ and for each $k \geq 0$, the inclusion (1) induces an isomorphism between k-th homotopy groups:

$$\pi_k(\mathrm{GL}_n(A)) \cong \pi_k(\mathrm{GL}_{n+1}(A)), \tag{2}$$

in other words, the inclusion map (1) is a homotopy equivalence for each $n \geq 1$. The irrational rotation algebra, the Cuntz algebras, and simple AF algebras are examples of K-stable algebras ([3], [4], [7]). Also for each finite dimensional compact metrizable space X, a $C(X)$-algebra (see [8, Appendix C]) A is K-stable if the fiber $A(x)$ is K-stable for each $x \in X$ ([5]). For each such algebra A, we have an isomorphism

$$\pi_k(\mathrm{GL}_n(A)) \cong \begin{cases} K_0(A) & k \ \text{ is odd}, \\ K_1(A) & k \ \text{ is even}, \end{cases}$$

for each $n \geq 1$.

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Two $*$-homomorphisms $\phi, \psi : A \to B$ between C^*-algebras A and B are said to be *homotopic*, written as $\varphi \simeq \psi$, if there exists a one-parameter family $(\phi_t : A \to B)_{t\in[0,1]}$ of $*$-homomorphisms (called a *homotopy*) such that $\phi_0 = \varphi, \phi_1 = \psi$, and for each $a \in A$, the map $[0,1] \to B$ defined by

$$t \mapsto \phi_t(a)$$

is continuous. Two C^*-algebras A and B are said to be *homotopy equivalent* if there exist $*$-homomorphisms $\varphi : A \to B, \psi : B \to A$ such that $\varphi \circ \psi \simeq \mathrm{id}_B, \psi \circ \varphi \simeq \mathrm{id}_A$. The K-stability of a C^*-algebra is a homotopy invariant property: if A and B are homotopy equivalent, then A is K-stable if and only if B is K-stable.

Topological shape theory, originated by K. Borsuk, studies topological spaces under the notion of *shape equivalence*. It is aimed to study spaces that exhibit "wild" local behavior from homotopy-theoretic view point. Two topological spaces are homotopy equivalent, then they are always shape equivalent, but the converse does not hold in general (see [2]). The topological shape theory has been most successful for the class of compact metrizable spaces. Seeking a non-commutative analogue of topological shape theory, Blackadar [1] systematically developed shape theory for separable C^*-algebras. The theory reduces to topological shape theory for compact metrizable spaces when restricted to commutative unital C^*-algebras. As in topological shape theory, two C^*-algebras A and B are shape equivalent in the sense of [1] if they are homotopy equivalent.

The present note records an observation that the K-stability is invariant under shape equivalence. The proof is very simple and straightforward, yet the fact seems unnoticed in the literature. The author hopes that the observation enlarges the class of K-stable C^*-algebras and gives an insight to the non-stable K-theory ([7]).

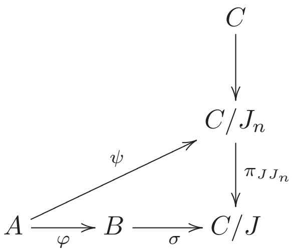
2 Result

First we recall the notion of shape equivalence of [1]. Throughout all C^*-algebras and all $*$-homomorphisms are assumed to be unital for simplicity. The standard modification described in [7] allows us to extend the result for general (not necessarily unital) C^*-algebras. All ideals of C^* algebras are assumed to be closed and two-sided. For two ideals I, J of a C^*-algebra A with $I \subset J$, $\pi_{IJ} : A/I \to A/J$ denotes the canonical projection.

Definition 2.1. *A $*$-homomorphism $\varphi : A \to B$ is said to be semiprojective if, for each C^*-algebra C and for each increasing sequence (J_n) of ideals with $\overline{\cup_n J_n} = J$ and for each $*$-homomorphism $\sigma : B \to C/J$, there exist an integer n and a $*$-homomorphism $\psi : A \to C/J_n$ such that $\sigma \circ \varphi =$*

$\pi_{J_n J} \circ \psi$. *If* ψ *can always be chosen as a* $*$*-homomorphism* $A \to C$, *then* φ *is said to be projective.*

A C^**-algebra* A *is said to be projective (resp. semiprojective) if* id_A *is projective (resp. semiprojective).*



For a compact metrizable space X, the commutative C^*-algebra $C(X)$ is semiprojective, with all algebras involved in the above definition being unital commutative C^*-algebras, if and only if X is an absolute neighborhood retract (ANR) [1, Proposition 2.11]. This indicates that the notion of semiprojective C^*-algebras is a non-commutative analogue of that of ANR spaces. The next theorem corresponds to the fact that every compact metrizable space is the limit of an inverse sequence of compact ANR's [2].

Theorem 2.2. *[1, Theorem 4.3] For every separable (unital)* C^**-algebra* A, *there exists an inductive sequence of* C^**-algebras and* $*$*-homomorphisms*

$$A_1 \xrightarrow{\gamma_1} A_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{i-1}} A_i \xrightarrow{\gamma_i} A_{i+1} \longrightarrow \cdots \tag{3}$$

such that $A \cong \varinjlim_i (A_i, \gamma_i)$ *and each* γ_i *is semiprojective.*

In the above sequence, for integers i, j with $i < j$, let γ_{ij} be the composition given by $\gamma_{ij} = \gamma_{j-1} \circ \cdots \circ \gamma_i : A_i \to A_j$. The above theorem allows us to introduce the notion of shape equivalence as follows.

Definition 2.3. *Let* A, B *be two* C^**-algebras and let* $(A_i, \gamma_i : A_i \to A_{i+1})$ *and* $(B_i, \delta_i : B_i \to B_{i+1})$ *be inductive sequences as in (3) such that* $A \cong \varinjlim_i (A_i, \gamma_i)$ *and* $B \cong \varinjlim_i (B_i, \delta_i)$. *We say that* A *and* B *are shape equivalent if there exist subsequences* $(m_k), (n_k)$ *of positive integers and* $*$*-homomorphisms* $(\alpha_k : A_{m_k} \to B_{n_k})_k$, $(\beta_k : B_{n_k} \to A_{m_k+1})_k$ *such that*

$$\begin{aligned} \beta_k \circ \alpha_k &\simeq \gamma_{m_k m_{k+1}}, \\ \alpha_{k+1} \circ \beta_k &\simeq \delta_{n_k n_{k+1}} \end{aligned} \tag{4}$$

for each k.

$$\begin{array}{ccc} A_{m_k} & \xrightarrow{\gamma_{m_k m_{k+1}}} & A_{m_{k+1}} \\ {\scriptstyle \alpha_k}\downarrow & \nearrow {\scriptstyle \beta_k} & \downarrow {\scriptstyle \alpha_{k+1}} \\ B_{n_k} & \xrightarrow[\delta_{n_k n_{k+1}}]{} & B_{n_{k+1}} \end{array}$$

It follows from [1, Theorem 4.8] that the above condition does not depend on the choice of the inductive sequences (A_i, γ_i) and (B_i, δ_i). Now our result is stated as follows.

Main Theorem. *Let A, B be two unital shape equivalent C^*-algebra. Then A is K-stable if and only if B is K-stable.*

We start with the next lemma.

Lemma 2.4. *Let $(\varphi_t : A \to B)_{0 \le t \le 1}$ be a one-parameter family that gives a homotopy between φ_0 and φ_1, and define $\Phi : A \times [0,1] \to B$ by*

$$\Phi(a,t) = \varphi_t(a), \ (a,t) \in A \times [0,1].$$

Then the map Φ is continuous.

Proof. Assume that a sequence $\{(a_n, t_n)\}$ of points of $A \times [0,1]$ converges to a point $(a,t) \in A \times [0,1]$. We have

$$\begin{aligned} \|\Phi(a_n,t_n) - \Phi(a,t)\| &\le \|\Phi(a_n,t_n) - \Phi(a,t_n)\| + \|\Phi(a,t_n) - \Phi(a,t)\| \\ &= \|\varphi_{t_n}(a_n) - \varphi_{t_n}(a)\| + \|\varphi_{t_n}(a) - \varphi_t(a)\| \\ &\le \|a_n - a\| + \|\varphi_{t_n}(a) - \varphi_t(a)\|, \end{aligned}$$

where the last inequality follows from the fact that each $*$-homomorphism is norm non-increasing. By the hypothesis, the last term converges to zero as $n \to \infty$ and we see that the sequence $\{\Phi(a_n, t_n)\}$ converges to $\Phi(a,t)$. □

The above lemma implies that the map $\mathrm{GL}_n(\Phi) : \mathrm{GL}_n(A) \times [0,1] \to \mathrm{GL}_n(B)$; $((a_{ij}), t) \mapsto (\varphi_t(a_{ij}))$ $((a_{ij}) \in \mathrm{GL}_n(A))$, is continuous for each positive integer n.

Proof of Main Theorem. Let A, B be two shape equivalent C^*-algebras and let $(A_i, \gamma_i), (B_i, \delta_i)$ be inductive sequences as in (3) with $A = \varinjlim(A_i, \gamma_i), B = \varinjlim(B_i, \delta_i)$. There exist sequences $(m_k), (n_k)$ of positive integers and $*$-homomorphisms $(\alpha_k : A_{m_k} \to B_{n_k})_k$, $(\beta_k : B_{n_k} \to A_{m_k+1})_k$ required in (4). Passing to the j-th homotopy groups with the aid of

Lemma 2.4, we obtain the following commutative diagram, where the induced homomorphisms are indicated by the same symbol for simplicity:

$$\begin{array}{ccccc}
\pi_j(\mathrm{GL}_n(A_{m_k})) & \xrightarrow{\gamma_{m_k m_{k+1}}} & \pi_j(\mathrm{GL}_n(A_{m_{k+1}})) & \xrightarrow{\gamma_{m_{k+1}\infty}} & \varinjlim_k \pi_j(\mathrm{GL}_n(A_{m_k})) \\
\downarrow \alpha_k^n & \nearrow \beta_k^n & \downarrow \alpha_{k+1}^n & & \downarrow \varinjlim_k \alpha_k^n \\
\pi_j(\mathrm{GL}_n(B_{n_k})) & \xrightarrow[\delta_{n_k n_{k+1}}]{} & \pi_j(\mathrm{GL}_n(B_{n_{k+1}})) & \xrightarrow[\delta_{n_k \infty}]{} & \varinjlim_k \pi_j(\mathrm{GL}_n(B_{n_k}))
\end{array}$$

We can directly verify that the limit homomorphism $\varinjlim_k \alpha_k^n : \varinjlim_k \pi_j(\mathrm{GL}_n(A_{n_k})) \to \varinjlim_k \pi_j(\mathrm{GL}_n(B_{n_k}))$ is an isomorphism of groups with the inverse homomorphism $\varinjlim_k \beta_k^n$.

The inclusions $i_{n,k}^A : \mathrm{GL}_n(A_{m_k}) \hookrightarrow \mathrm{GL}_{n+1}(A_{m_k})$ and $i_{n,k}^B : \mathrm{GL}_n(B_{n_k}) \hookrightarrow \mathrm{GL}_{n+1}(B_{n_k})$ induce homomorphisms of homotopy groups. Passing to the inductive limit, we obtain homomorphisms $\varinjlim_k i_{n,k}^A : \varinjlim_k \pi_j(\mathrm{GL}_n(A_{m_k})) \to \varinjlim_k \pi_j(\mathrm{GL}_{n+1}(A_{m_k}))$ and $\varinjlim_k i_{n,k}^B : \varinjlim_k \pi_j(\mathrm{GL}_n(B_{n_k})) \to \varinjlim_k \pi_j(\mathrm{GL}_{n+1}(B_{n_k}))$. It is straightforward to verify that the following diagram is commutative:

$$\begin{array}{ccc}
\varinjlim_k \pi_j(\mathrm{GL}_n(A_{m_k})) & \xrightarrow{\varinjlim_k i_{n,k}^A} & \varinjlim_k \pi_j(\mathrm{GL}_{n+1}(A_{m_k})) \\
\downarrow \varinjlim_k \alpha_k^n & & \downarrow \varinjlim_k \alpha_k^{n+1} \\
\varinjlim_k \pi_j(\mathrm{GL}_n(B_{n_k})) & \xrightarrow[\varinjlim_k i_{n,k}^B]{} & \varinjlim_k \pi_j(\mathrm{GL}_{n+1}(B_{n_k}))
\end{array}$$

Since $\varinjlim_k \alpha_k^n$ and $\varinjlim_k \alpha_k^{n+1}$ are both isomorphisms, we see that

(∗) $\varinjlim_k i_{n,k}^A$ is an isomorphism if and only if $\varinjlim_k i_{n,k}^B$ is an isomorphism.

By [6, Proposition 1.4], we have an isomorphism $\pi_j(\mathrm{GL}_n(A)) \cong \varinjlim_k \pi_j(\mathrm{GL}_n(A_{m_k}))$ in such a way that the diagram below is commutative:

$$\begin{array}{ccc}
\pi_j(\mathrm{GL}_n(A)) & \xrightarrow{i_n^A} & \pi_j(\mathrm{GL}_{n+1}(A)) \\
\downarrow \cong & & \cong \downarrow \\
\varinjlim_k \pi_j(\mathrm{GL}_n(A_{m_k})) & \xrightarrow{\varinjlim_k i_{n,k}^A} & \varinjlim_k \pi_j(\mathrm{GL}_{n+1}(A_{m_k}))
\end{array}$$

where i_n^A denotes the homomorphism induced by the inclusion. The same holds for the induced homomorphism $i_n^B : \pi_j(\mathrm{GL}_n(B)) \to \pi_j(\mathrm{GL}_{n+1}(B))$. Combining these with (∗), we see that i_n^A is an isomorphism if and only if i_n^B is an isomorphism, for each $n \geq 1$ and for each $k \geq 0$.

This proves Main Theorem.

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