ソボレフ流の時間無限大における体積集中について(*)

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Abstract

We study a doubly nonlinear parabolic equation describing the gradient flow associated with the Sobolev inequality, called as $p-$Sobolev flow. We show that the asymptotic behavior of the $p-$Sobolev flow at time-infinity is characterized by the so-called volume and energy concetration phenomenon.

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1 Introduction

In this report we consider the following doubly nonlinear degenerate and singular parabolic equation, called $p-$Sobolev flow,

$$(1.1)\qquad \begin{cases} \partial_t u^q - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda(t)u^q & \text{in } \Omega_\infty := \Omega \times (0,\infty) \\ \|u(t)\|_{q+1} = 1 & \text{for } t \geq 0 \\ u = u_0 & \text{on } \partial_p\Omega_\infty := \partial\Omega \times (0,\infty) \end{cases}$$

Here Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$, $p > 1$, $p \leq q+1 \leq p^*$ with $p^* := \frac{np}{n-p}$ if $1 < p < n$ and any finite positive number if $p \geq n$, $u = u(x,t)$ is a nonnegative function defined for $(x,t) \in \Omega_\infty$, $\nabla_\alpha = \partial/\partial x_\alpha$, $\alpha = 1,\ldots,m$, $\nabla u = (\nabla_\alpha u)$ is the spatial gradient of a function u, $|\nabla u|^2 = \sum_{\alpha=1}^m (\nabla_\alpha u)^2$ and $\partial_t u$ is the derivative on time t. The initial and boundary data $u_0 = u_0(x)$ is in the Sobolev space $\mathrm{W}_0^{1,p}(\Omega)$, nonnegative, bounded and $\|u_0\|_{q+1} = 1$. The function $\lambda(t)$ is computed by the condition $(1.1)_2$ as follows: Multiply the equation in $(1.1)_1$ by u

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and integrate the resulting one by parts on space to have

$$\frac{d}{dt}\frac{q}{q+1}\|u(t)\|_{q+1}^{q+1}+\|\nabla u(t)\|_p^p=\lambda(t)\|u(t)\|_{q+1}^{q+1}\Longrightarrow\lambda(t)=\|\nabla u(t)\|_p^p,$$

where $\|f\|_p$ is the $L^p(\Omega)-$norm of a function f. The system above describes the negative directed gradient flow in the constrained extremal problem for the $p-$energy. The corresponding Euler-Lagrange equation is given as the $p-$Laplace type equation, which has only trivial solution if the domain Ω is star-shaped with the origin. This fact is verified by a Pohožaev type identity and Hopf's maximum principle, which are proved through the regularized $p-$Laplace equation (see [5]). Thus, a solution of the evolution equation may have concentration points of volume, local $(q+1)-$powered integral, at infinite time, by the volume conservation $\|u(t)\|_{q+1}=1$. Our main purpose is to study such asymptotic behavior of a solution to the evolution equation above.

The first result is the global existence a weak solution of (1.1) and its regularity (see [6, 7]).

Theorem 1 (A global existence and regularity) *Let $p>1$ and $p\le q+1\le p^*$. Suppose that u_0 belongs to $\mathrm{W}_0^{1,p}(\Omega)$, is nonnegative, bounded, $\|u_0\|_\infty<\infty$, and $\|u_0\|_{q+1}=1$. Then, there exists a global weak solution $u\in C\left([0,\infty);L^{q+1}(\Omega)\right)\cap L^\infty(0,\infty;W_0^{1,p}(\Omega))$ of the Cauchy-Dirichlet problem (1.1), satisfying the energy inequalities*

$$\|u(t)\|_{q+1}=1,\qquad\forall t\ge 0,\tag{1.2}$$

$$\|\partial_t u^{\frac{q+1}{2}}\|^2_{\mathrm{L}^2(\Omega_\infty)}+\sup_{0<t<\infty}E(u(t))\le E(u_0),\tag{1.3}$$

where $E(u):=\|\nabla u\|_p^p/p$ is the $p-$energy of u. Moreover, the solution u is positive and bounded, $0<u(t,x)\le e^{pE(u_0)t/q}\|u_0\|_\infty$ for any $(t,x)\in\Omega_\infty$, and u and its spatial gradient ∇u are locally in time-space continuous in Ω_∞.

We shall study the asymptotic behavior around infinite time of the global solution to (1.1) obtained in Theorem 1. The global solution of (1.1) strongly or weakly converges to a limit function in $W^{1,p}_{\mathrm{loc}}(\Omega)$ along a time-sequence increasingly tending to ∞ and the limit function is naturally a weak solution of the stationary equation corresponding to $(1.1)_1$. In the case of weak convergence, further, there may appear the so-called energy and volume gap at infinite time, leading to energy and volume concentration.

The asymptotic profile at a concetration point of the global soluton of (1.1) is shown in the following result. Applying the concentration-compactness result, we obtain a characterization of concentration of volume and energy on a microscopic scale, where the term *microscopic* is borrowed from the result [12, Lemma 3.4, p.72] for the case $p = 2$. To state the result, let ρ_0 be a fixed small positive number. Let $\eta = \eta(x)$ be a smooth function on $\mathbb{R}^n$ such that $\eta = 0$ outside $B(0, \rho_0)$ and $\eta = 1$ on $B(0, \rho_0/2)$.

Theorem 2 (Concentration-compactness) *Let $\frac{2n}{n+2} < p < n$ and $q + 1 = p^*$. Let $\{t_k\}$, $t_k \nearrow \infty$ and $\{r_k\}$, $r_k \searrow 0$ as $k \to \infty$. There exist a subsequence $\{t_k\}$ (non-relabelled), an integer N, $N-$points $\{x_i\} \subset \Omega$, subsequence $\{r_{k,i}\}$ and a sequence $L_{k,i} \nearrow \infty$ as $k \to \infty$, $i = 1, \ldots, N$, such that the follwoing convergence holds true:*

$$u(x, t_k) - \sum_{i=1}^{N} L_{k,\,i}\,(\eta\, v)\left(L_{k,\,i}^{\frac{q+1-p}{p}}\,(x - x_i)\right) \longrightarrow u_\infty(x)$$

$$\text{strongly and locally in } W^{1,p} \cap L^{q+1}(\mathbb{R}^n)\,(k \to \infty), \tag{1.4}$$

where v is a positive and bounded weak solution of $-div(|\nabla v|^{p-2}\nabla v) = \lambda_\infty v^q$ in $\mathbb{R}^n$ with a positive constant λ_∞, and v and its gradient ∇v are locally continuous in $\mathbb{R}^n$. Moreover, the volume and energy decompositions appear at time infinity: As $k \to \infty$,

$$\mathrm{V}(u(t_k)) \longrightarrow \mathrm{V}(u_\infty) + N\widetilde{\mathrm{V}}(v), \tag{1.5}$$

$$\mathrm{E}(u(t_k)) \longrightarrow \mathrm{E}(u_\infty) + N\widetilde{\mathrm{E}}(v), \tag{1.6}$$

where we put

$$\mathrm{V}(u) = \int_\Omega u^{q+1}\,dx, \qquad \widetilde{\mathrm{V}}(u) = \int_{\mathbb{R}^n} u^{q+1}\,dx,$$

$$\mathrm{E}(u) = \int_\Omega |\nabla u|^p\,dx, \qquad \widetilde{\mathrm{E}} = \int_{\mathbb{R}^n} |\nabla u|^p\,dx.$$

We shall explain the implication of Theorem 2. For this purpose we recall some results on the asymptotic convergence of the Palais-Smale like sequence. In the Laplacian case $p = 2$ we have the global compactness result established by Struwe ([14]). The result was extended to the $p-$Laplacian case for $1 < p < n$ (see [10, 11, 3]). Theorem 2 establishs the so-called volume and energy equalities and thus, completely characterizes the asymptotic behavior as infinity-time of the nonnegative solutions to (1.1). See [14, Proposition 2.1, p. 513], [10, Theorem 1.2, pp. 471-472] for the case $1 < p < n$.

The limit function v at time-infinity in Theorem 2 is given as the extremal function attaining the best constant in the Sobolev inequality, called Talenti function. Refer to [15] for the Laplacian case, [13, 2, 17] for the $p-$Laplacian case.

2 Preliminary estimate

We present the local boundedness available for a nonnegative weak solution to (1.1), obtained in Theorem 1. This is the key estimation for showing the volume concentration at the limit as time tends to ∞ of a solution of (1.1).

Lemma 3 (Local boundedness) *Let $1 < p < n$ and $q + 1 = p^*$. Suppose that u is a nonnegative weak solution to (1.1), obtained in Theorem 1. Let r_0 be a positive number satisfying $r_0 E(u_0) \le 1$ and $Q(r_0) \equiv B(x_0, r_0) \times (t_0 - (r_0)^p, t_0) \subset \Omega_\infty$. Put $\gamma = \frac{p(n+2)}{n}$. There exists a positive constant $\hat{\delta}_0 = \hat{\delta}_0(n, p, q) \le 1$ such that the following holds true: For any positive number $\delta_0 \le \hat{\delta}_0$, there exists a positive number k_0 such that*

$$(2.1) \qquad k_0 \ge \frac{1}{r_0^{n+p} \delta_0^\gamma}, \qquad 1 = \frac{1}{\delta_0} \left(\frac{1}{r_0^{n+p} k_0} + \frac{1}{|\hat{Q}|} \int_{\hat{Q}} \frac{u^{q+1}}{k_0^{q+1}} \, dx \, dt \right)^{1/\gamma},$$

where $\hat{Q}(k_0, r_0) \equiv B\left(x_0, k_0^{(p-(q+1))/p} r_0\right) \times (t_0 - r_0^p, t_0)$, and there holds

$$(2.2) \qquad u(x_0, t_0) \le 4k_0.$$

The proof of Lemma 3 is based on De Giorgi's type local energy estimates for truncated solutions, of which the detail will be appeared in a fothcoming paper. Here we shall show how to determine the local boundedness constant, of which the way is intrinsic to a solution and may be of its own interest. We emphasize that the equation (2.1) corresponds to (2.3) in the following proposition.

Proposition 4 (Intrinsic local boundedness) *Let $r_0 > 0$ and $\delta_0 \in (0, 1)$. Let $Q(r_0) = B(x_0, r_0) \times (t_0 - (r_0)^p, t_0) \subset \Omega_\infty$. Put $\beta = \frac{p(q-1)}{n}$ and $\gamma = \frac{p(n+2)}{n}$ (so that $\beta + \gamma = q + 1 = p^*$). Then there is a unique positive real number k_0 such that if $u \in L^{q+1}(Q(r_0))$ and $u \ge 0$, then there is a unique solution k_0, $k_0 \ge r_0^{-n-p} \delta_0^{-\gamma}$, to the equation*

$$(2.3) \qquad k_0 = \frac{1}{\delta_0} \left(\frac{k_0^{-1+\gamma}}{r_0^{n+p}} + \fint_{\hat{Q}(k_0, r_0)} \frac{u^\beta}{k_0^\beta} u^\gamma \, dx \, dt \right)^{1/\gamma},$$

where $\hat{Q}(k_0, r_0) = B\left(x_0, k_0^{(p-(q+1))/p} r_0\right) \times (t_0 - (r_0)^p, t_0)$. Moreover, the root satisfies $k_0 \equiv k(u, r_0, \delta_0) \nearrow \infty$ as $r_0 \searrow 0$ or $\delta_0 \searrow 0$.

Proof. Since

$$\fint_{\hat{Q}(k_0,r_0)} \frac{u^\beta}{k_0^\beta} u^\gamma \, dx\, dt = \frac{k_0^{-\beta+\frac{n}{p}(q+1-p)}}{r_0^{n+p}|B(0,1)|} \int_{\hat{Q}(k_0,r_0)} u^{\beta+\gamma}\, dx\, dt$$

and

$$-\beta + \frac{n}{p}(q+1-p) = \gamma - q - 1 + \frac{n}{p}(q+1) - n = \gamma + \frac{n-p}{p}(q+1) - n = \gamma\,,$$

we have that

$$\left(\frac{k_0^{-1+\gamma}}{r_0^{n+p}} + \fint_{\hat{Q}(k_0,r_0)} \frac{u^\beta}{k_0^\beta} u^\gamma \, dx\, dt \right)^{1/\gamma}$$

$$= k_0 \left(\frac{1}{r_0^{n+p}} \left[\frac{1}{k_0} + \frac{1}{|B(0,1)|} \int_{\hat{Q}(k_0,r_0)} u^{q+1}\, dx\, dt \right] \right)^{1/\gamma} . \tag{2.4}$$

The function

$$k_0 \mapsto h(k_0)\,, \qquad h(k_0) = \frac{1}{r_0^{n+p}} \left[\frac{1}{k_0} + \frac{1}{|B(0,1)|} \int_{\hat{Q}(k_0,r_0)} u^{q+1}\, dx\, dt \right],$$

is continuous and strictly decreasing function of k_0 and $h(k_0) \downarrow 0$ as $k_0 \uparrow \infty$ for any given $r_0 > 0$. Moreover $h({r_0}^{-n-p}) \geq 1$. Therefore there must be a unique $k_0^* > r_0^{-n-p}$ such that

$$h(k_0^*) = \delta_0^\gamma\,.$$

It is easy to see that this root converges to infinity as r_0 or δ_0 tends to zero. This proves the claim.

□

By the use of Lemma 3 we show the uniform local boundedness for solutions of (1.1).

Lemma 5 (Uniform boundedness) *Let $1 < p < n$ and $q+1 = p^*$. Suppose that u is a nonnegative weak solution to (1.1), obtained in Theorem 1. Suppose that, for some positive numbrs ε_0, $r_0 \leq 1$ and $Q(r_0) = B(x_0, r_0) \times (t_0 - r_0^p, t_0) \subset \Omega_\infty$,*

$$\fint_{t_0-r_0^p}^{t_0} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,\,r_0))}\, dt \leq \varepsilon_0. \tag{2.5}$$

Then there holds, for positive numbers $r_0' = r_0'(r_0, \varepsilon_0)$, $C = C(r_0, \varepsilon_0)$ *and* $Q(r_0') = B(x_0, r_0') \times (t_0 - (r_0')^p, t_0)$,

$$\sup_{(x,\,t)\in Q(r_0')} |u| \le C. \tag{2.6}$$

Proof. Let $(x_0', t_0') \in Q(r_0/2)$ be an arbitrarily taken and fixed. We shall employ Lemma 3 for the proof, where r_0 and (x_0, t_0) are replaced by $r_0/2$ and (x_0', t_0'), respectively. Clearly, $Q'(r_0/2) \equiv B(x_0', r_0/2) \times (t_0' - (r_0/2)^p, t_0')$ is contained in $Q(r_0)$. Let k_0 be chosen as in (2.1) of Lemma 3. From $\dfrac{n(p-(q+1))}{p} = -\dfrac{np}{n-p} = -(q+1)$ and (2.5), it follows that, letting $\hat{Q}'(k_0, r_0/2) \equiv B\left(x_0', k_0^{(p-(q+1))/p}(r_0/2)\right) \times \left(t_0' - (r_0/2)^p, t_0'\right)$,

$$\fint_{\hat{Q}'(k_0, r_0/2)} \frac{u^{q+1}}{k_0^{q+1}}\, dx\, dt = \frac{2^{n+p}}{r_0^{n+p}|B(1)|} \int_{Q(r_0)} u^{q+1}\, dx\, dt \le \frac{2^{n+p}\varepsilon_0}{r_0^n |B(1)|}, \tag{2.7}$$

where we note that $k_0 \ge 1$ by $\delta_0 \le 1$ and $r_0 \le 1$. Choosing $k_0' \ge k_0$ so large that

$$\frac{1}{\delta_0^\gamma}\left(\frac{1}{r_0^{n+p}k_0'} + \frac{2^{n+p}\varepsilon_0}{r_0^n|B(1)|}\right) \quad \text{is very close to} \quad \frac{2^{n+p}\varepsilon_0}{\delta_0^\gamma r_0^n |B(1)|}, \tag{2.8}$$

we obtain from (2.2) in Lemma 3 that

$$u(x_0', t_0') \le 4k_0' \quad \text{for any } (x_0', t_0') \in Q(r_0/2). \tag{2.9}$$

Here we notice the dependence of k_0', $k_0' = k_0'(r_0, \delta_0, \varepsilon_0, n, p)$, and thus, the assertion (2.6) follows from (2.9), letting $r_0' = (k_0')^{\frac{p-(q+1)}{p}}(r_0/2) (\le r_0/2)$. □

3 Concentration phenomenon

We shall present the proof of Theorem 2. For this purpose we prepare two lemmata.

First we shall show the asymptotic behavior of the global solution of (1.1), obtained in Theorem 1. This controls the global solution of (1.1) at infinite time and leads to deriving the asymptotic convergence to a limit function which is a stationary solution corresponding to (1.1).

Lemma 6 (ε−strong compactness) *Let* $\frac{2n}{n+2} < p < n$ *and* $q+1 = p^*$. *Let* $\{t_k\}$, $t_k \nearrow \infty$ *as* $k \to \infty$. *There exist subsequence* $\{t_k\}$ *(non-relabelled), a positive number* $\epsilon_0 > 0$ *and at most finite* *points* $\{x_1, \ldots, x_N\} \subset \Omega$, $0 < N < \infty$, *such that there holds for any positive number* $r \leq 1$ *and all* $i = 1, \ldots, N$,

$$\liminf_{k \nearrow \infty} \fint_{t_k - r^p}^{t_k} \|u(t)\|_{L^{q+1}(B(x_i, r))}^{q+1} \, dt \geq \varepsilon_0 \tag{3.1}$$

and the sequence $\{u(t_k)\}$ *is strongly convergent in the Sobolev space* $W^{1,p}$ *on any compact subset of* $\Omega \setminus \{x_1, \ldots, x_N\}$.

Proof. First we notice that the conditions, $\frac{2n}{n+2} < p < n$ and $q + 1 = p^*$, imply that $q \geq 1$. Let u be nonnegative weak solution to (1.1), obtained in Theorem 1.

We shall show the following: There exists a sequence of times $\{\tau_k\}$, $\tau_k \nearrow \infty$ as $k \to \infty$ such that the sequence of solutions $\{u(\tau_k)\}$ converges to a weak solution of the stationary equation corresponding to (1.1).

First we take a subsequence $\{t''_k\}$ of $\{t'_k\}$ such that $t''_{k+1} - t''_k \geq 1$ for all $k = 1, \ldots$ and $t''_k \nearrow \infty$ as $k \to \infty$. Write as $I(k) = \big(t''_k, t''_{k+1}\big)$, $k = 1, \ldots$. Now we prove that there exists a sequence $\{\tau_k\}$ such that $\tau_k \in I(k)$, $k = 1, \ldots$, $\tau_k \nearrow \infty$ as $k \to \infty$ and

$$\lim_{k \to \infty} \int_\Omega |\partial_t u^q(\tau_k)| \, dx = 0. \tag{3.2}$$

Indeed, by (1.3) in Theorem 1 there holds

$$\sum_{k=1}^{\infty} \fint_{I(k)} \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dx \, dt \leq \int_0^\infty \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dx \, dt < \infty,$$

where we use that the length $|I(k)|$ of $I(k)$ is larger than 1 by the choice of t''_k. From the mean-value theorem, for each $k = 1, \ldots$ there exists a number $\tau_k \in I(k)$ such that, as $k \to \infty$,

$$\int_\Omega \left|\partial_t u^{\frac{q+1}{2}}(\tau_k)\right|^2 dx \leq \fint_{I(k)} \int_\Omega \left|\partial_t u^{\frac{q+1}{2}}\right|^2 dx \, dt \to 0.$$

For $q \geq 1$ the chain rule of weak differential enables us to compute as

$$\partial_t u^q = \frac{2q}{q+1} u^{\frac{q-1}{2}} \partial_t u^{\frac{q+1}{2}}, \tag{3.3}$$

since the function $z^{\frac{2q}{q+1}}$ is locally Lipschitz on $z \in [0, \infty)$. The fact above and the Hölder inequality yield the estimation

$$\int_\Omega |\partial_t u^q(\tau_k)|\,dx \le \frac{2q}{q+1}\|u^{\frac{q-1}{2}}(\tau_k)\|_2\|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2$$

$$\le \frac{2q}{q+1}|\Omega|^{\frac{1}{q+1}}\sup_{t\in(0,\,\infty)}\|u(t)\|_{q+1}^{\frac{q-1}{2}}\|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2$$

$$\le \|u_0\|_{q+1}^{\frac{q-1}{2}}\|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2 \longrightarrow 0$$

as $k\to\infty$, which gives (3.2).

Next we claim that the integral equation

$$\int_\Omega \left(\partial_t u^q(\tau_k)\phi + |\nabla u|^{p-2}\nabla u(\tau_k)\cdot\nabla\phi - (\lambda u^q)(\tau_k)\phi\right)\,dx = 0 \tag{3.4}$$

holds true for any $\phi\in C_0^\infty(\Omega)$.

Let $0<\varepsilon, h\searrow 0$ and define a cut-off function on time $\eta_h=\eta_h(t)$ such that η_h is Lipschitz on $\mathbb{R}$, $\eta_h=1$ in $[\tau_k-\varepsilon+h,\,\tau_k+\varepsilon-h]$, $\eta_h=0$ in $\mathbb{R}\setminus(\tau_k-\varepsilon,\,\tau_k+\varepsilon)$ and $|\partial_t\eta_h|\le 1/h$ in $\mathbb{R}$. Then, we use the test function $\phi\eta_h$ in the weak form of $(1.1)_1$. Noting the integrability of each term appearing in the resulting equality, by the Lebesgue convergence theorem we pass to the limit as $h\searrow 0$ and have

$$\int_{\tau_k-\varepsilon}^{\tau_k+\varepsilon}\left\{\int_\Omega \left(\partial_t u^q\phi + |\nabla u|^{p-2}\nabla u\cdot\nabla\phi - (\lambda u^q)\phi\right)\,dx\right\}dt = 0$$

and then, dividing the both side of the above equation by 2ε, from the Lebesgue's differenrtial theorem available for integrable functions we can take the limit as $\varepsilon\searrow 0$ in the resulting equation to obtain the claim (3.4).

From (1.2) and (1.3) in Theorem 1 we see that the sequence $\{u(\tau_k)\}$ is bounded in $W^{1,p}(\Omega)$ and thus, by the compactness of Sobolev empbedding into the Lebesgue space we have a (non-relabeled) subsequence $\{\tau_k\}$, the limit function $w\in W_0^{1,p}(\Omega)$ and a finite number λ_∞ such that, as $k\to\infty$,

$$\begin{aligned} &u(\tau_k)\longrightarrow w && \text{weakly in } W^{1,p}(\Omega),\\ &u(\tau_k)\longrightarrow w && \text{strongly in } L^r(\Omega),\ \ \forall r\in[1,p^*), \text{ and almost everywhere } \Omega,\\ &\lambda(\tau_k)\longrightarrow\lambda_\infty, \end{aligned} \tag{3.5}$$

where we use Mazur's theorem verifying that the closed subspace $W_0^{1,p}(\Omega)$ of $W^{1,p}(\Omega)$ is weakly closed in $W^{1,p}(\Omega)$.

We also have the following strong convergence of gradients: There exists a (non-relabeled) subsequence $\{u(\tau_k)\}$ such that

$$\nabla u(\tau_k) \longrightarrow \nabla w \qquad \text{strongly in } L^r(\Omega), \quad \forall r \in [1, p), \tag{3.6}$$

of which the proof is referred in [9, Lemma 5.3, p. 19, Appendix E, p. 43].
By means of the convergences (3.2), (3.5) and (3.6) we have the identity holding true for any $\phi \in C^\infty(\Omega)$

$$\int_\Omega \left(|\nabla w|^{p-2} \nabla w \cdot \nabla \phi - \lambda_\infty w^q \phi \right) dx = 0. \tag{3.7}$$

Further we can verify that the sequence $\{u(\tau_k)\}$ strongly converges to the limit w in $W^{1,p}(\Omega \setminus \mathcal{N})$ for some set of finitely many points $\mathcal{N} = \{x_1, \ldots, x_N\}$. In fact we shall demonstrate the convergence

$$\nabla u(\tau_k) \longrightarrow \nabla w \qquad \text{strongly in } L^p_{\mathrm{loc}}(\Omega \setminus \mathcal{N}). \tag{3.8}$$

For the proof we shall employ the local boundedness of the solution to (1.1).
Fix $x_0 \in \Omega$ and assume that for some positive $r_0 \le 1$ there holds

$$\liminf_{k \nearrow \infty} \fint_{t_k - r_0^p}^{t_k} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,\, r_0))}\, dt < \varepsilon_0.$$

Then we choose a subsequence $\{t'_k\}$ of $\{t_k\}$ such that

$$\fint_{t'_k - r_0^p}^{t'_k} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,\, r_0))}\, dt \le \varepsilon_0. \tag{3.9}$$

Applying Lemma 5 with (3.9), we have positive numbers $r'_0 = r'_0(r_0, \varepsilon_0)$, $C = C(r_0, \varepsilon_0)$ and $Q(r_0; x_0, t'_k) \equiv B(x_0,\, r'_0) \times (t'_k - (r'_0)^p,\, t'_k)$ such that

$$\sup_{Q(r_0; x_0, t'_k)} |u| \le C, \tag{3.10}$$

yielding the uniform boundedness in $B(x_0, r_0)$ of the solutions $\{u(t'_k)\}$, $k = 1, \ldots$.

Next we shall show the validity of the following convergences as $k \to \infty$:

$$\int_\Omega \partial_t u(\tau_k) \, (u(\tau_k) - w) \, dx \longrightarrow 0, \tag{3.11}$$

$$\int_{B(x_0, r'_0/2)} (u^q(\tau_k) - w^q) \, (u(\tau_k) - w) \, dx \longrightarrow 0. \tag{3.12}$$

Noting (3.3) again we estimate as

$$\begin{aligned}\int_\Omega |\partial_t u(\tau_k)||u(\tau_k)-w|\,dx &\le \frac{2q}{q+1}\int_\Omega |\partial_t u^{\frac{q+1}{2}}(\tau_k)|\left(u^{\frac{q+1}{2}}(\tau_k)+u^{\frac{q-1}{2}}(\tau_k)|w|\right)dx \\ &\le \frac{2q}{q+1}\|\partial_t u^{\frac{q+1}{2}}(\tau_k)\|_2\left(\|u(\tau_k)\|_{q+1}^{\frac{q+1}{2}}+\|u(\tau_k)\|_{q+1}^{\frac{q-1}{2}}\|w\|_{q+1}\right), \end{aligned} \tag{3.13}$$

where the Hölder inequality is used in the second inequality with $q \ge 1$. Thus, the convergence (3.11) follows from (3.3).

For the proof of (3.12) we note the strong integral convergence as

$$u(\tau_k) \longrightarrow w \qquad \text{strongly in } L^\gamma(B(x_0,r_0')) \text{ for any finite } \gamma \ge 1, \tag{3.14}$$

by the use of the convergence $(3.5)_2$ and the uniform boundedness (3.10). By the Hölder inequality we simply estimate and take the limit as $k \to \infty$ in the resulting inequality as

$$\begin{aligned}\int_{B(x_0,r_0')} |u^q(\tau_k)-w^q|\,|u(\tau_k)-w|\,dx &\le \int_{B(x_0,r_0')} \left(|u(\tau_k)|^q+|w|^q\right)|u(\tau_k)-w|\,dx \\ &\le \left(\|u(\tau_k)\|_{q+1}^q+\|w\|_{q+1}^q\right)\|u(\tau_k)-w\|_{L^{q+1}(B(x_0,r_0'))} \longrightarrow 0,\end{aligned}$$

where the convergence (3.14) is used in the last line. The validity of (3.12) is shown.

Here we recall the algebraic inequalities as follows (Refer the proof in [1, Lemma 2.2] and [4, inequality (24)]) : There holds for any vectors $P, Q \in \mathbb{R}^n$ that

$$\begin{aligned}&\left(|P|^{p-2}P-|Q|^{p-2}Q\right)\cdot(P-Q) \ge C_1\left(|P|+|Q|\right)^{p-2}|P-Q|^2, \\ &\left||P|^{p-2}P-|Q|^{p-2}Q\right| \le C_2\left(|P|+|Q|\right)^{p-2}|P-Q|.\end{aligned} \tag{3.15}$$

Now we subtract (3.7) from (3.4) and use the test function $\eta^2(u(\tau_k)-w)$ in the resulting equation, where the function $\eta=\eta(x)$ is Lipschitz on $\mathbb{R}^n$ such that $\eta=1$ in $B(x_0,r_0'/2)$, $\eta=0$ outside $B(x_0,r_0')$ and $|\nabla\eta| \le 2/r_0'$. By the use of (3.15) we have, if $p>2$,

$$\begin{aligned}&C'\int_{B(x_0,r_0')} \eta^2|\nabla u(\tau_k)-\nabla w|^p\,dx \\ &\le C\int_{B(x_0,r_0')} |\nabla\eta|^2\left(|\nabla u(\tau_k)|+|\nabla w|\right)^{p-2}|u(\tau_k)-w|^2\,dx \\ &\quad -\int_{B(x_0,r_0')} \eta^2\partial_t u(\tau_k)\left(u(\tau_k)-w\right)dx+\lambda_\infty\int_{B(x_0,r_0')} \eta^2\left(u^q(\tau_k)-w^q\right)\left(u(\tau_k)-w\right)dx\end{aligned}$$

$$(3.16)\qquad + (\lambda(\tau_k) - \lambda_\infty) \int_{B(x_0, r_0')} \eta^2 u^q(\tau_k)\,(u(\tau_k) - w)\,dx$$

$$\longrightarrow 0 \qquad \text{as } k \to \infty,$$

where we use the convergences (3.11), (3.12) and $(3.5)_3$.
If $1 < p < 2$ we use (3.15) to have the inequality

$$\begin{aligned}
&\int_{B(x_0,r_0')} \eta^2\,|\nabla u(\tau_k) - \nabla w|^p\,dx \\
&\le \left(\int_{B(x_0,r_0')} \eta^2\,(|\nabla u(\tau_k)| + |\nabla w|)^{p-2}\,|\nabla u(\tau_k) - \nabla w|^2\,dx \right)^{\frac{p}{2}} \\
&\qquad\qquad \times \left(\int_{B(x_0,r_0')} \eta^2\,(|\nabla u(\tau_k)| + |\nabla w|)^p\,dx \right)^{\frac{2-p}{2}} \\
&\le C \left(\int_{B(x_0,r_0')} \eta^2 \left(|\nabla u(\tau_k)|^{p-2}\nabla u(\tau_k) - |\nabla w|^{p-2}\nabla w\right) \cdot (\nabla u(\tau_k) - \nabla w)\,dx \right)^{\frac{p}{2}} \\
&\qquad\qquad \times \left(\int_{B(x_0,r_0')} \eta^2\,(|\nabla u(\tau_k)| + |\nabla w|)^p\,dx \right)^{\frac{2-p}{2}}.
\end{aligned}$$

At this stage we evaluate the last term in the above inequality. The integral term in the 1st brace is equal to the same as 3rd one in (3.16) and the integral term in the 2nd brace is bounded by (1.3) in Theorem 1. Thus, from the same reasoning as (3.16) this last term converges to 0 as $k \to \infty$. Therefore we have that $\nabla u(\tau_k)$ converges to ∇w strongly in $L^p(B(x_0, r_0'/2))$ as $k \to \infty$. The convergence (3.8) follows from by a usual covering argument with the strong convergence of gradients above.

The finiteness of concetration points $\mathcal{N}$ in (3.1) is verfied as follows: We compute as

$$\begin{aligned}
\sum_{i=1}^{N} \int_{t_k'-r_0^p}^{t_k'} \|u(t)\|_{L^{q+1}(B(x_i,\,r_0))}^{q+1}\,dt &= \int_{t_k'-r_0^p}^{t_k'} \sum_{i=1}^{N} \|u(t)\|_{L^{q+1}(B(x_i,\,r_0))}^{q+1}\,dt \\
&= \int_{t_k'-r_0^p}^{t_k'} \|u(t)\|_{L^{q+1}\left(\cup_{i=1}^{N} B(x_i,\,r_0)\right)}^{q+1}\,dt
\end{aligned}$$

$$\leq \fint_{t'_k - r_0^p}^{t'_k} \|u(t)\|_{L^{q+1}(\Omega)}^{q+1}\, dt = 1,$$

where we use (1.2) in the last line. Taking the limitinf on $k \nearrow \infty$ in both side of the above inequality yields the estimation

$$\begin{aligned} N\varepsilon_0 &\leq \sum_{i=1}^{N} \liminf_{k\nearrow\infty} \fint_{t'_k - r_0^p}^{t'_k} \|u(t)\|_{L^{q+1}(B(x_i,\, r_0))}^{q+1}\, dt \\ &\leq \liminf_{k\nearrow\infty} \sum_{i=1}^{N} \fint_{t'_k - r_0^p}^{t'_k} \|u(t)\|_{L^{q+1}(B(x_i,\, r_0))}^{q+1}\, dt \leq 1 \end{aligned}$$

and thus,

$$N \leq \frac{1}{\varepsilon_0}. \tag{3.17}$$

The proof of Lemma 6 is completed. □

The next result yields the asymptotic profile around a concetration point at infinite time of the global soluton of (1.1).

Lemma 7 (Volume and energy concentration) *Let $\frac{2n}{n+2} < p < n$ and $q + 1 = p^*$. Let $\{t_k\}$, $t_k \nearrow \infty$ and $\{r_k\}$, $r_k \searrow 0$ as $k \to \infty$. There exist a positive number $\epsilon_0 > 0$, an integer N, $N-$points $\{x_i\} \subset \Omega$, subsequences $\{t_{k,i}\}$, $\{r_{k,i}\}$ and a sequence of positive numbers $L_{k,i} \nearrow \infty$ as $k \to \infty$, $i = 1, \ldots, N$, such that the followings hold for each x_i, $i = 1, \ldots, N$: For brevity, letting $x' = x_i$, $t_k = t_{k,i}$, $r_k = r_{k,i}$ and $L_k = L_{k,i}$,*

$$\liminf_{k\nearrow\infty} \|u(t_k)\|_{L^{q+1}(B(x',\, r_k))}^{q+1} \geq \varepsilon_0;$$

$$v_k(x) := L_k^{-1} u\left(x' + L_k^{\frac{p-(q+1)}{p}} x,\ t_k\right) \longrightarrow v(x)$$

$$\text{strongly and locally in } W^{1,p} \cap L^{q+1}(\mathbb{R}^n)\ (k \to \infty), \tag{3.18}$$

where v is a positive and bounded weak solution of $-div(|\nabla v|^{p-2}\nabla v) = \lambda_\infty v^q$ in $\mathbb{R}^n$ with a positive constant λ_∞, and v and its gradient ∇v are locally continuous in $\mathbb{R}^n$.

Proof. Let $x_0 = x_i$, $i = 1, \ldots, N$ be any point where (3.1) holds true for any positive $r \leq 1$. Let $\{t_k\}$, $t_k \nearrow \infty$ as $k \to \infty$. Let $\{r_l\}$ be a sequence of postive numbers $r_l \searrow 0$ as $l \to \infty$. Then, by

(3.1) we have, for any r_l, $l = 1, \ldots,$

$$\liminf_{k\nearrow\infty} \fint_{t_k-(r_l)^p}^{t_k} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,\,r_l))}\,dt \geq \varepsilon_0 \tag{3.19}$$

and from the mean-value theorem there exists a number t_{kl}, $t_k-(r_l)^p < t_{kl} < t_k$ for each $k, l = 1, \ldots$ such that

$$\|u(t_{kl})\|^{q+1}_{q+1} = \fint_{t_k-(r_l)^p}^{t_k} \|u(t)\|^{q+1}_{L^{q+1}(B(x_0,\,r_l))}\,dt. \tag{3.20}$$

By Cantor's diagonal argument, (3.19) and (3.20) we can take subsequences $\{r'_k\}$ of $\{r_l\}$ and $\{t'_{kk}\}$ of $\{t_{kl}\}$ such that $t_k - (r'_k)^p < t'_{kk} < t_k$ and

$$\begin{aligned} &t'_{kk} \nearrow \infty, \qquad r'_k \searrow 0 \qquad \text{as } k \to \infty, \\ &\liminf_{k\nearrow\infty} \|u(t'_{kk})\|^{q+1}_{L^{q+1}(B(x_0,\,r'_k))} \geq \frac{\varepsilon_0}{2}. \end{aligned} \tag{3.21}$$

Let us write $\{t'_{kk}\}$ as $\{t_k\}$ and $\{r'_k\}$ as $\{r_k\}$.

Hereafter we shall fix $k = 1, \ldots$ and write as $t_0 = t_k$ and $r_0 = r_k$. Let $Q(r_0) = B(x_0, r_0) \times (t_0 - (r_0)^p, t_0)$. Let $(x'_0, t'_0) \in Q(r_0)$ be arbitrarily taken and be fixed. Make a local prabolic cylinder $Q'(r_0) = B(x'_0,\, r_0) \times (t'_0 - (r_0)^p,\, t'_0)$ with vertex at (x'_0, t'_0). We now employ Lemma 3 in $Q'(r_0)$. Thus, we have positive numbers $\delta_0 \leq 1$ and L' such that

$$\begin{aligned} &L' \geq (r_0)^{-n-p}\delta_0^{-\gamma}, \\ &\hat{Q}'(L', r_0) = B\left(x'_0, (L')^{(p-(q+1))/p} r_0\right) \times \left(t'_0 - (r_0)^p,\, t'_0\right), \end{aligned}$$

$$1 = \frac{1}{\delta_0}\left(\frac{1}{(r_0)^{n+p}L'} + \frac{1}{|\hat{Q}'(L', r_0)|}\int_{\hat{Q}'(L', r_0)} \frac{u^{q+1}}{(L')^{q+1}}\,dx\,dt\right)^{1/\gamma}, \tag{3.22}$$

$$u(x'_0, t'_0) \leq 4L'. \tag{3.23}$$

Here, in $(3.22)_3$ and (3.23), the positive number L' may depend on (x'_0, t'_0). Now we claim that the positive numbers L' is bounded uniformly on (x'_0, t'_0). Indeed there exists a positive $L > L'$ such that

$$L \geq (2r_0)^{-n-p}\delta_0^{-\gamma},$$

$$\frac{1}{(\delta_0)^\gamma}\left(\frac{1}{(r_0)^{n+p}L} + \frac{1}{|Q(r_0)|}\int_{Q(2r_0)} u^{q+1}\,dx\,dt\right) < \frac{2}{(\delta_0)^\gamma|Q(r_0)|}\int_{Q(2r_0)} u^{q+1}\,dx\,dt \tag{3.24}$$

Because the positive constant L in (3.24) does not depend on any $(x_0', t_0') \in Q$. In fact, for any positive $l \geq 1$ and any point $(x_0', t_0') \in Q(r_0)$, $\hat{Q}'(l, r_0)$ is contained in $Q(2r_0)$ and thus, we have,

$$\frac{1}{\delta_0}\left(\frac{1}{(r_0)^{n+p}l} + \frac{1}{|\hat{Q}'(l, r_0)|}\int\limits_{\hat{Q}'(l,r_0)} \frac{u^{q+1}}{l^{q+1}}\,dx\,dt\right)^{1/\gamma}$$

$$= \frac{1}{\delta_0}\left(\frac{1}{(r_0)^{n+p}l} + \frac{1}{|B(1)|r_0^{n+p}}\int\limits_{\hat{Q}'(l,r_0)} u^{q+1}\,dx\,dt\right)^{1/\gamma}$$

$$< \frac{1}{\delta_0}\left(\frac{1}{(r_0)^{n+p}l} + \frac{1}{|Q(r_0)|}\int\limits_{Q(2r_0)} u^{q+1}\,dx\,dt\right)^{1/\gamma}. \tag{3.25}$$

L' is a root of the equation (3.22) and L is that of the equation: $1 =$ the left hand side of (3.24), and thus, L doed not dpend on $(x_0', t_0') \in Q$ and $L' < L$.
Therefore from (3.23) and the observation above it follows that

$$u(x_0', t_0') \leq 4L \qquad \text{for any } (x_0', t_0') \in Q. \tag{3.26}$$

We write L as L_k to indicate the dependence of L on t_k and r_k. Now we introduce the scaled solution defined as

$$v_k(x,t) = \frac{u\left(x_0 + L_k^{\frac{p-(q+1)}{p}}x,\ t_k + t\right)}{L_k}, \tag{3.27}$$

$$(x,t) \in Q'(k) = B(k) \times J(k), \qquad B(k) = B\left(0,\ r_k L_k^{\frac{q+1-p}{p}}\right), \quad J_k = (-(r_k)^p,\, 0)\,.$$

From $(3.24)_1$, the space-width of $Q'(k)$ is computed as

$$r_k L_k^{\frac{q+1-p}{p}} \geq (\delta_0)^{-\frac{p\gamma}{n-p}}(r_k)^{\frac{-p(1+n+p)-n}{n-p}} \nearrow \infty \quad \text{as } k \to \infty, \tag{3.28}$$

because $\frac{-p(1+n+p)-n}{n-p} < 0$ and δ_0 is a fixed positive number, and the time-length $(r_k)^p \searrow 0$ as $k \to \infty$, and thus, the sequence of sets $\{Q'(k)\}$ converges to all of space $\mathbb{R}^n$. By (3.26), we have the boundedness

$$\sup_{Q'(k)} v_k \leq 4 \tag{3.29}$$

and compute the integral quantities of v_k for any $t \in (-(r_k)^p,\ 0)\,,$ as

$$\|v_k(t)\|_{L^{q+1}(B(k))} = \|u(t_k + t)\|_{L^{q+1}(B(x_0, r_k))}, \tag{3.30}$$

(3.31) $$\|\nabla v_k(t)\|_{L^p(B(k))} = \|\nabla u(t_k + t)\|_{L^p(B(x_0, r_k))},$$

(3.32) $$\int_{-(r_k)^p}^{0} \|\partial_t v_k^{\frac{q+1}{2}}(t)\|^2_{L^2(B(k))}\, dt = \int_{t_k-(r_k)^p}^{t_k} \|\partial_t u^{\frac{q+1}{2}}(t)\|^2_{L^2(B(x_0, r_k/2))}\, dt.$$

By virtue of the boundedness (1.2) and (1.3) we can argue similarly as (3.2)-(3.7) to have subsequences $\{t'_k\}$, $\{r'_k\}$ (non-relabelled), a sequence $\{\tau_k\}$ and the limit $v \in W^{1,p} \cap L^{q+1}(\mathbb{R}^n)$ such that

(3.33)

$$t'_{k+1} - t'_k \geq 1, \qquad \tau_k \in \left(t'_k - (r'_k)^p,\, t'_k\right) \qquad \text{for all } k = 1, \ldots,$$

$$\int_{B(k)} \left|\partial_t v_k^{\frac{q+1}{2}}(\tau_k)\right|^2 dx \longrightarrow 0 \qquad \text{as } k \to \infty,$$

$$\int_{\mathbb{R}^n} \left(\partial_t v_k^q(\tau'_k)\phi + |\nabla v_k(\tau_k)|^{p-2}\nabla v_k(\tau_k)\cdot\nabla\phi - (\lambda v_k^q)(\tau_k)\right) dx = 0 \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n),$$

$$v_k(\tau_k) \longrightarrow v \qquad \text{weakly in } W^{1,p}(\mathbb{R}^n),$$

$$v_k(\tau_k) \longrightarrow v \qquad \text{strongly in } L^r(\mathbb{R}^n),\quad \forall r \in [1, p^*), \text{ and almost everywhere } \mathbb{R}^n,$$

$$\lambda(\tau_k) \longrightarrow \lambda_\infty,$$

$$\nabla v_k(\tau_k) \longrightarrow \nabla v \qquad \text{strongly in } L^r(\mathbb{R}^n),\quad \forall r \in [1, p),$$

$$\int_{\mathbb{R}^n} \left(|\nabla v|^{p-2}\nabla v\cdot\nabla\phi - \lambda_\infty v^q\phi\right) dx = 0 \qquad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n).$$

Further we have the strong convergence of gradients

(3.34) $$\nabla v_k(\tau_k) \longrightarrow \nabla v \qquad \text{strongly and locally in } L^p(\mathbb{R}^n),$$

of which the proof is performed similarly as in (3.11)-(3.16) by the use of (3.29).
Finally, the limit function v is a non-negative bounded weak solution of the stationary equation on $\mathbb{R}^n$ as $(3.33)_8$, where the boundedness follows from the boundedness (3.29) and the almost everywhere convergence $(3.33)_5$, and thus, v and its gradient ∇v are locally continuous in $\mathbb{R}^n$ by the regularity of the $p-$Laplace equation. The limit v is not identically zero in $\mathbb{R}^n$ from the non-vanishing local-volume (3.21) and the strong convergence $(3.33)_5$ and thus, v is positive in $\mathbb{R}^n$ by the strong maximum principle for the $p-$Laplace operator in [16].
The proof of Lemma 7 is completed. □

Proof of Theorem 2. Now we shall give the proof of Theorem 2.
The validity of (1.4) follows from (3.18) in Lemma 7, where we choose the time-sequence $\{t_k\}$ as the common subsequence of $\{t_{k,i}\}$, $i = 1, \ldots, N$. Any boundary concentration never appears because the limit function around a concentration point on boundary $\partial\Omega$ is a positive stationary solution with zero Dirichlet boundary condition in the half space $\mathbb{R}^n_+$, which, however should be trivial from the Liouville type result in [10, Theorem 1.1, pp. 470-471], where the strong maximum principle takes an important role (refer to [16]). Thus, the concentration points are only in the interior of the domain Ω.
The volume and energy concentrations (1.5) and (1.6) is shown similarly as in [10, Proof of Theorem 1.2, Sect. 4]. Here we notice that the solutions $\{u(t_k)\}$ of (1.1) is just a Palais-Smale like sequence for the functional $(1/p)\mathrm{E}(u) + (1/(q+1))\lambda_\infty \mathrm{V}(u)$ on $W_0^{1,p}(\Omega)$, which is verified in the proof of Lemma 6. □

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